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CONTENTS

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HYPERSPHERICAL GONIOMETRY; AND ITS APPLICATION TO CORRELATION THEORY FOR N VARIABLES.

BY JAMES McMAHON*, Cornell University, Ithaca, N.Y.

THE chief suggestion for this essay is contained in the following quotation from an article by Professor Karl Pearson in *Biometrika* (Vol. XI, p. 237):

"It is greatly to be desired that the 'trigonometry' of higher dimensioned space should be fully worked out; for all our relations between multiple correlation and partial correlation coefficients of n variates are properties of the 'angles,' 'edges' and 'perpendiculars' of spheropolyhedra in multiple space. It would be a fine task for a pure mathematician to write a treatise on spherical polyhedrometry; he need not fear that his results would be without practical application, for they embrace the whole range of problems from anatomy to medicine, and from medicine to sociology and ultimately to the doctrine of evolution."

It is hoped that the following essay will serve as a beginning in the directions just indicated, both for mathematical theory and for practical application. In Part I the formulae of spherical trigonometry are generalized for the hypersphere in n dimensions, the underlying hypergeometry being first sketched in a form adapted to the purpose in hand. In Part II a far-reaching connexion between the geometry of the hypersphere and the theory of correlation for n variables is established by linearly transforming the "ellipsoids" of equal frequency (on an n-dimensional correlation chart) into a certain family of hyperspherical surfaces. The generalized formulae of hyperspherical goniometry are then applied to furnish easy proofs of various old and new theorems in multiple and partial correlation, including the relations alluded to above, and to point the way to further developments.

PART I.

GONIOMETRY OF HIGHER DIMENSIONED SPACES.

1. *Spaces of various orders.* Every space considered in this essay is euclidean; this implies that its elements are points and straight lines, and that it contains every point of the indefinite straight line connecting any pair of its points; also that the axioms and postulates which Euclid enunciated for spaces of two and of three dimensions are to be regarded as holding for the higher dimensioned spaces considered. A space of n-dimensions will be denoted by S_n.

* On account of the sudden and untimely death of Professor James McMahon, the manuscript for this paper was left by him in an unfinished condition. It has been put in final form for publication by his colleague, Professor F. W. Owens, who has attempted to preserve as far as possible the intent of the author.

*Just as an S_3 is generated by the lines connecting the points of a given plane (an S_2) with a given point outside of it, so an S_4 is thought of as generated by the lines connecting points of a given S_3 with a given point outside of that space, and as containing all the points of the line passing through any pair of its points. As a definite S_2 can be assigned by means of three given points not in the same straight line (S_1), and a definite S_3 by means of four given points not in the same S_2 and no three of which are in the same S_1, so a convenient way of assigning a definite S_4 is by means of five given points which are not in the same S_3, no four of which are in the same S_2, and no three in the same S_1. Five such points are said to determine an S_4, since any two S_4's which contain these five points will coincide throughout. An S_5 is thought of as generated by the lines connecting points of a given S_4 with a given point outside of that space.

Proceeding thus, step by step, we say that an S_n is generated by the lines connecting points of a given S_{n-1} with a given point outside of that space; and a determinate S_n can be assigned by means of $n+1$ given points which are not in the same S_{n-1}, and no r of which are in the same S_{r-2}, for any two S_n's which contain these $n+1$ points will coincide throughout.

The same S_n can also be defined by means of the n lines connecting any one of these $n+1$ points with the remaining n points. When an S_n is defined in this way by n concurrent lines, it is requisite that no r of them lie in the same S_{r-1}; e.g. four concurrent lines of which three lie in the same S_2 do not uniquely determine an S_4 but only an S_3, and many four spaces could contain them.

2. *The space common to given spaces.* It is sometimes important to know whether two given spaces have any points, lines, or spaces in common. We have to generalize such familiar statements as the following: An S_1 and another S_1 which lie in the same S_3 but not in the same S_2 have no point common; an S_1 and an S_2 which lie in the same S_3 but not in the same S_2 have just one point common. These are cases of the following more general principle: An S_p and an S_q which lie in the same S_n but not in the same S_{n-1} have an S_{p+q-n} in common (but not a space of higher order) when $p+q>n$; they have no point in common when $p+q<n$; and just one point in common when $p+q=n$.

For clearness we take definite values of n, p, q; and the general method of proof will be sufficiently indicated by showing that an S_4 and an S_5 which lie in the same S_7 but not in the same S_6 have an S_2 in common but not an S_3. For, suppose they have a common S_3, and let it be defined by the four points A, B, C, D; let the S_4 be defined by the five points A, B, C, D, E; and the S_5 by the six points A, B, C, D, F, G; then the S_4 and the S_5 lie in the S_6 defined by the seven points A, B, C, D, E, F, G, contrary to the hypothesis; hence the S_4 and the S_5 have not an S_3 in common. Next, suppose they have an S_1 in common but not an S_2, and let the common S_1 be defined by the two points A, B; let the given S_4 be defined by the five points A, B, C, D, E; and the S_5 by the six points $A, B, F, G,$

* From this point on to the end of Art. 2 all the statements concerning lines, planes, spaces, etc., must be interpreted projectively, i.e. every two lines in a plane intersect in a point, etc.

H, I, then the S_4 and the S_5 both lie in the S_8 defined by the nine points $A, B, C, D, E, F, G, H, I$, but not in the same S_7, contrary to the hypothesis. *A fortiori* two other suppositions fail, viz. that they have only one point in common, and that they have no point in common; hence they have an S_2 in common but not an S_3.

A quite similar method of exclusion can be applied in the proof of the remaining two parts of the theorem stated above.

By successive applications of this theorem it follows that an S_p, an S_q, and an S_r, which lie in the same S_n but not in the same S_{n-1}, have in common a space of order $p+q+r-2n$ but no higher; and more generally, that an S_{p_1}, an S_{p_2}, an S_{p_3}, ..., and an S_{p_r}, which lie in the same S_n but not in the same S_{n-1}, have in common a space of order $p_1+p_2+\ldots+p_r-(r-1)n$ (but no higher) when this is a positive number, and just one point in common when this number is zero.

Cor. In an n-space, a line (S_1) meets a hyperplane (S_{n-1})* in a single point; a plane (S_2) meets a hyperplane in a line; a plane does not meet a line when $n=4$ unless they lie in the same S_3; a plane meets a plane in a single point when $n=4$ unless the two planes are in the same S_3 in which case they meet in a line; two hyperplanes meet in an S_{n-2}; and r hyperplanes meet in an S_{n-r}; thus $n-1$ hyperplanes meet in a line, and n hyperplanes meet in a single point; but a line does not meet an S_{n-2} unless they lie in the same S_{n-1}.

3. *Parallel lines.* Two lines in an S_n are said to be *parallel* if they lie in the same S_2 and if they do not meet however far prolonged. Through a given point A a line can be drawn parallel to a given line L by the usual construction performed in the S_3 determined by L and A, and there is only one such parallel in a "euclidean space."

(a) If a line L and a point A are in any S_r, then the parallel to L through A lies in the same S_r; for it lies in the S_2 determined by L and A, and this S_2 is situated in the S_r.

(b) If two lines L and M through O are respectively parallel to two lines L' and M' through O', then the two pairs are in the same S_3 and include equal angles; for L and M and the point O' determine an S_3, hence the parallel to L through O' lies in this S_3, and so does the parallel to M through O', as shown in (a); it follows that the pairs include equal angles by a theorem in three-dimensional geometry.

4. *An n-dimensioned parallelopiped and an n-dimensional co-ordinate system.* Let an S_n be defined by $n+1$ given points; this implies that no n of them lie in the same S_{n-1} (Art. 2). Take any one of these points as origin O, and let the lines through O and the remaining n points be denoted by $OL_1, OL_2, \ldots, OL_n$. On OL_1 lay off any length $OA_1 = a_1$; and translate OA_1 parallel to itself so that each point moves in a direction parallel to OL_2 through a given distance a_2, then OA_1 will generate a parallelogram in the plane of OL_1, OL_2; and the co-ordinates of the four vertices with regard to the axes OL_1 and OL_2 are

$$(a_1, a_2), \quad (0, a_2), \quad (a_1, 0), \quad (0, 0).$$

* For reasons which appear later, an S_{r-1} belonging to an S_r is called a *hyperplane* of the S_r.

Then if we translate the parallelogram so that each of its points moves parallel to the direction OL_3 through a given distance a_3 it will generate a parallelopiped in the S_3 determined by the lines (OL_1, OL_2, OL_3); this translation leaves the x_1 and the x_2 co-ordinates unchanged, but we have to annex the third co-ordinate $x_3 = a_3$ in the new position of the parallelogram and $x_3 = 0$ in its first position, hence the parallelopiped will have the eight vertices whose co-ordinates are

$$(a_1, a_2, a_3), \quad (0, a_2, a_3), \quad (a_1, 0, a_3), \quad (0, 0, a_3),$$
$$(a_1, a_2, 0), \quad (0, a_2, 0), \quad (a_1, 0, 0), \quad (0, 0, 0).$$

If we next translate this parallelopiped so that each of its points moves parallel to the fourth line OL_4 (which is not in the S_3 defined by the first three lines) through a given distance a_4, it will generate a four-dimensioned parallelopiped in the S_4 determined by (OL_1, OL_2, OL_3, OL_4); this translation leaves the x_1, x_2, x_3 co-ordinates of a point unchanged, but we have to annex the fourth co-ordinate $x_4 = a_4$ in the new position of the parallelopiped and $x_4 = 0$ in its old position; thus the four-dimensioned parallelopiped generated by the translation of the three-dimensioned one has the sixteen vertices

$$(a_1, a_2, a_3, a_4), \quad (0, a_2, a_3, a_4), \quad \ldots, \quad (0, 0, a_3, a_4), \quad \ldots, \quad (0, 0, 0, a_4), \quad \ldots, \quad (0, 0, 0, 0).$$

Continuing thus we obtain the co-ordinates of the 2^n vertices of the n-dimensioned parallelopiped which has the n axes of co-ordinates for adjacent sides. There are $2n$ faces arranged in opposite pairs.

The face $x_1 = 0$ is the S_{n-1} determined by the $n-1$ lines $OL_2, OL_3, \ldots, OL_n$, and is a hyperplane of the S_n defined by the n lines $OL_1, OL_2, \ldots, OL_n$, and it is denoted by

$$P_1 \equiv [L_2, L_3, \ldots, L_n].$$

Similarly the hyperplane $x_r = 0$ is denoted by

$$P_r \equiv [L_1, L_2, \ldots, (L_r), \ldots, L_n],$$

and the opposite face $x_r = a_r$ is the hyperplane defined by the $n-1$ lines drawn through the vertex

$$V \equiv (a_1, a_2, \ldots, a_r, \ldots, a_n)$$

parallel to the $n-1$ axes $L_1, L_2, \ldots, (L_r), \ldots, L_n$.

Since V is obtained by successive translation of O parallel to the axes through distances $a_1, a_2, \ldots, a_n$, respectively, it is evident that the vector OV equals the vector sum of n vectors drawn parallel to the axes and of lengths $a_1, a_2, \ldots, a_n$, respectively.

5. *Projection of a vector upon a line. Condition of perpendicularity.* Using the same axes as before, let a vector of length r connecting the origin with the point $V \equiv (x_1, x_2, \ldots, x_n)$ be projected on a line OA that makes the angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ with the respective axes; then, by projecting on OA the vectors mentioned in Art. 4, we get

$$r \cos\theta = x_1 \cos\alpha_1 + x_2 \cos\alpha_2 + \ldots + x_n \cos\alpha_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where θ is the angle between OV and OA.

When OV and OA are at right angles, we have the condition of perpendicularity

$$x_1 \cos \alpha_1 + x_2 \cos \alpha_2 + \ldots + x_n \cos \alpha_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

This may be regarded as the equation of the locus generated by all the lines passing through O perpendicular to the given line OA whose direction angles are $\alpha_1, \alpha_2, \ldots, \alpha_n$. This locus will be shown in Art. 7 to be an S_{n-1}, that is, a hyperplane.

6. *Common perpendicular to $n-1$ lines defining an S_{n-1}.* We can use equation (2) to prove the following principle:

If any n concurrent lines define an S_n, then to any $n-1$ of these lines can be drawn, in the same S_n, one and but one common perpendicular through their common point. For, the condition that the line OV is perpendicular to each of the given lines furnishes $n-1$ equations like (2); and from these we can solve for the $n-1$ ratios $x_1 : x_2 : \ldots : x_n$, and obtain a result in the form

$$\frac{x_1}{c_1} = \frac{x_2}{c_2} = \ldots = \frac{x_n}{c_n},$$

which determines the direction of the common perpendicular OV.

Cor. To n given lines defining an S_n there is no common perpendicular lying in the same S_n.

7. *Common perpendicular to r lines defining an S_r.* It is known that in an S_3 a line perpendicular to two lines defining an S_2 is perpendicular to every line in the S_2, and is then said to be perpendicular to the S_2. We wish to prove the more general principle that in an S_n a line perpendicular to r concurrent lines defining an S_r is perpendicular to every line in the S_r, and it will then be said to be perpendicular to the S_r. We shall first prove that if this theorem be true for a definite value of r, then it will remain true when r is increased by unity. Let the $r+1$ lines defining the S_{r+1} be $L_1, L_2, \ldots, L_r, L_{r+1}$ meeting at O, and let L be any other line through O in the same S_{r+1}; pass an S_2 through L_{r+1} and L, and let it meet the S_r defined by $L_1, L_2, \ldots, L_r$, in the line M lying in the S_{r+1}. (In an S_{r+1} any S_r and an S_2 have an S_1 in common by Art. 2.) Then, by hypothesis, a line perpendicular to the lines $L_1, L_2, \ldots, L_r$ is perpendicular to every line in the same S_r and is hence perpendicular to M; therefore any line drawn perpendicular to the lines $L_1, L_2, \ldots, L_r, L_{r+1}$ is perpendicular to both M and L_{r+1}, and hence to L which is in the same S_2 with these two lines. Thus the principle holds for S_{r+1}, provided it holds for S_r, but it holds for $r=2$, hence for $r=3, 4, \ldots$, and so on, indefinitely.

Note. A perpendicular to an S_r does not lie in the same S_r (Art. 6, Cor.). It will be shown in Art. 9 that all the perpendiculars to an S_r, drawn in the same S_n, lie in an S_{n-r}.

8. *The perpendicular to an S_{n-1}. Equation of an S_{n-1}.* It follows from Arts. 6, 7 that the single line OL drawn perpendicular to the $n-1$ lines which determine an S_{n-1} is perpendicular to every line in the latter; and it is then said to be perpendicular to the S_{n-1}. Conversely any line OM perpendicular to OL lies in the

S_{n-1}; for suppose OM does not lie in the S_{n-1}, then the S_2 through OL and OM meets the S_{n-1} in an S_1 (Art. 2), and this line is perpendicular to OL, thus there could be two perpendiculars to OL in the same S_2, which is impossible. Hence all the perpendiculars to OL lie in the same S_{n-1}. It follows from Art. 5 that the equation of any S_{n-1} passing through O is of the form

$$x_1 \cos \alpha_1 + x_2 \cos \alpha_2 + \ldots + x_n \cos \alpha_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the direction angles of its normal.

9. *The r equations of an S_{n-r}. Locus of normals to an S_r. Two perpendicular spaces.* The locus of two linear equations of the form (3) is an S_{n-2}, for an S_{n-1} and another S_{n-1} situated in the same S_n have a common S_{n-2}, any point of which satisfies both equations. Similarly the locus of r such independent linear equations is an S_{n-r} (Art. 2, Cor.); the locus of $n-1$ such equations is an S_1, which is thus the common line of $n-1$ hyperplanes situated in an S_n.

The locus of all the perpendiculars at O to an S_r, situated in an S_n, is an S_{n-r}; for, let the S_r be defined by r lines through O; then any common perpendicular at O to these lines is perpendicular to the S_r (Art. 7), and satisfies r linear equations like (3), hence, as above, the locus of all the perpendiculars is an S_{n-r}; and this S_{n-r} will be said to be normal to the S_r at the point O. It is evident that any line through O in either space is perpendicular to every line through O in the other space.

Two spaces so related that every line of either is perpendicular to every line of the other are said to be *completely normal* to each other. Thus an S_p determined by p of the normals at O is completely normal to the given S_r. An S_p containing part of these normals not sufficient to determine an S_p, but just sufficient to determine an S_q, is said to be *partially normal* to the S_r with an index of normality expressed by the fraction q/p. Thus an S_4 containing only sufficient of the normals at O to determine an S_3 is three-fourths normal to the S_r; and a plane through just one normal is half normal to the S_r at O. Two perpendicular planes may be either completely normal or half normal to each other. Completely normal spaces cannot have more than one point common. There can be completely normal spaces with no points common, but we do not need to consider them in this essay.

10. *The perpendicular from a point to a hyperplane; and from a point to any S_r. Projections.* The existence of the perpendicular from a given point to a given S_{n-1} rests on the following theorem: In any S_n if a line ON is perpendicular at O to an S_{n-1}, then any parallel line $O'N'$ is perpendicular to the same S_{n-1} at O'. To prove this, let $O'L'$ be any line in the S_{n-1} and let OL be parallel to $O'L'$, then OL lies in the S_{n-1} (Art. 3 (*a*)), and is perpendicular to ON (Art. 8), but the angles LON and $L'O'N'$ are equal (Art. 3 (*b*)), hence $O'N'$ is perpendicular to $O'L'$, which is any line through O' in the hyperplane, therefore $O'N'$ is perpendicular to the hyperplane.

Hence we can draw a perpendicular from N' by first erecting the perpendicular ON (Art. 8), and then passing $N'O'$ parallel to ON; and there can be only one

such perpendicular, otherwise there could be a triangle having the sum of its angles greater than two right angles.

The perpendicular from a point to any S_r is drawn in the same way, the space of operation being the S_{r+1} determined by the S_r and the given point; here $r+1$ takes the place of n. The foot of the perpendicular is called the *projection* of the given point on the S_r.

The projection of a line OL on a hyperplane P passing through O is defined as the line which is common to P and to the S_2 determined by OL and the normal ON. This S_2 contains the perpendicular from any point of OL (Art. 3 (a)), hence the projection of a line OL contains the projections of all its points.

The projection of a line OL on any S_r through O is defined in the same way, the space of operation being the S_{r+1} determined by the S_r and the given line; here $r+1$ takes the place of n.

11. *An n-dimensioned angle. Two conjugate angloids.* In a given S_n, the figure formed by n half-lines $L_1, L_2, \ldots, L_n$, issuing from a point O, and sufficient to define the S_n (implying that no r of them lie in the same S_{r-1}), will be called an *n-dimensioned angle*, or briefly an *angloid*, and the n half-lines will be called *rays*. The angloid will be indicated by

$$\alpha \equiv (L_1, L_2, \ldots, L_n),$$

and the hyperplane of the $n-1$ rays obtained by omitting L_r will be denoted by

$$P_r \equiv [L_1 L_2 \ldots (L_r) \ldots L_n].$$

The normal drawn at O in the S_n to the hyperplane P_r, towards the same side as L_r, will be denoted by L_r'. The angloid formed by the n normals so drawn, namely

$$\alpha' \equiv (L_1', L_2', \ldots, L_n'),$$

will be said to be conjugate to α. Any ray L_r' of α' is evidently perpendicular to any ray L_s of α when r is different from s. It is clear that the relation between the two angloids α and α' is reciprocal, that is, any ray L_r is normal to the hyperplane

$$P_r' \equiv [L_1' L_2' \ldots (L_r') \ldots L_n'].$$

That hyperplane of the $(n-1)$-space P_r which is obtained by omitting L_s is denoted by

$$P_{rs} \equiv [L_1 L_2 \ldots (L_r L_s) \ldots L_n].$$

This is an S_{n-2} and will be called a *planoid* of the original S_n; it is the common space (or intersection) of the hyperplanes P_r and P_s. Similarly P_{rs} has its own hyperplanes such as

$$P_{rst} \equiv [L_1 L_2 \ldots (L_r L_s L_t) \ldots L_n];$$

this is an S_{n-3} and is another planoid of the S_n; it is the common space of P_{rs} and of P_t, that is, of the three hyperplanes P_r, P_s, P_t.

It will be seen from Art. 9 that all the perpendiculars drawn at O to the r-planoid $(L_1 L_2 \ldots L_r)$ generate the $(n-r)$-planoid

$$P_{12\ldots r} \equiv (L'_{r+1} L'_{r+2} \ldots L'_n),$$

since the latter planoid is determined by the $n-r$ lines indicated, each of which is perpendicular to each of the r lines $L_1, L_2, \ldots, L_r$, hence any line of either planoid is perpendicular to any line of the other; and the two planoids are completely normal to each other. In particular all the perpendiculars to the planoid $P_{rs} \equiv [L_1 L_2 \ldots (L_r L_s) \ldots L_n]$ drawn in the given S_n lie in the plane $(L_r' L_s')$; and all the perpendiculars to the $(n-3)$-space P_{rst} lie in the three-space $(L_r' L_s' L_t')$ and *vice versa*.

12. *Dihedral angle of an angloid.* Just as in a three-dimensioned angle the dihedral angle between two of its planes is measured by the plane angle between the two lines drawn in these two planes perpendicular to their common line, so in the n-dimensioned angle α, the dihedral angle between two of its hyperplanes P_r and P_s is defined as measured by the plane angle between the two lines N_r and N_s drawn in these two hyperplanes at O perpendicular to their common S_{n-2}, which is the planoid P_{rs}, and towards the same sides of these hyperplanes as L_r and L_s are situated, respectively.

We shall first prove that this plane angle of the dihedral angle is equal to the supplement of the angle between the lines L_r' and L_s' drawn normal to these hyperplanes at O; for the two normals N_r and N_s both lie in the plane $(L_r' L_s')$ by the statement at the end of Art. 11, and N_r is perpendicular to L_r' since it lies in P_r, and N_s is perpendicular to L_s' since it lies in P_s. Moreover N_r is at the same side of L_r' as L_s' is, and N_s is at the same side of L_s' as L_r' is, hence, by an elementary principle in plane geometry,

$$(N_r, N_s) = \text{supp}\,(L_r', L_s'),$$

therefore dihedral
$$(P_r, P_s) = \text{supp}\,(L_r', L_s') \quad \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

similarly
$$(P_r', P_s') = \text{supp}\,(L_r, L_s) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

We shall next prove the following useful generalization:

In any S_n, the dihedral between two r-planoids situated in the same S_{r+1} equals the supplement of the dihedral between their respective normal $(n-r)$-planoids situated in the same S_{n-r+1}.

Let the two r-planoids be

$$(L_1 L_2 \ldots L_{r-1} L_r),\ (L_1 L_2 \ldots L_{r-1} L_{r+1})$$

which lie in the same S_{r+1}, namely $(L_1 L_2 \ldots L_{r+1})$, then their respective normal planoids are

$$(L'_{r+1} L'_{r+2} \ldots L'_n),\ (L'_r L'_{r+2} \ldots L'_n);$$

the dihedral between the latter planoids is measured by the plane angle between the lines N_p and N_q drawn at O in the respective planoids and normal to their common planoid $(L'_{r+2} L'_{r+3} \ldots L'_n)$, and hence both lying in the normal planoid $(L_1 L_2 \ldots L_{r+1})$; but N_p and N_q are evidently normal to the two r-planoids first mentioned, since they lie in their normal planoids, respectively, and they are in the same S_{r+1} with these r-planoids, hence their angle (N_p, N_q) is supplemental to the dihedral between the r-planoids, by the theorem proved above; therefore the two dihedrals are supplemental.

Cor. In particular we have

$$(P_{rstu}, P_{rstv}) = \text{supp}\,(L_r'L_s'L_t'L_u', L_rL_sL_tL_v') \quad \ldots\ldots\ldots\ldots\ldots(6),$$

the dihedral on the right being contained between two hyperplanes in the five-space $(L_r'L_s'L_t'L_u'L_v')$, and that on the left between two hyperplanes in the $(n-3)$-space P_{rst}.

13. *Inclination of a line to a hyperplane; and to any S_r.* The angle between a line and its projection on a hyperplane is called the *inclination* of the line to the hyperplane. It is evident from Art. 10 that it is the complement of the angle between the line OL and the normal at O to the hyperplane. Hence, in the notation of Art. 11, the inclination of the ray L_r to the corresponding hyperplane P_r is given by

$$(L_r, P_r) = \text{comp}\,(L_r, L_r') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

In the same way the inclination of a line OL to any S_r through O is defined as the angle between the line and its projection on the S_r, and we have

$$(L, S_r) = \text{comp}\,(L, N), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7a),$$

where N is the normal at O to the S_r drawn in the S_{r+1} determined by the S_r and the line OL; here the S_{r+1} just mentioned is the space of operation.

From this we derive the following useful theorem: The inclination of a line OL to any space S_r passing through O is the complement of the inclination of the same line to the normal space S'_{n-r}.

To prove this, let N be the normal at O to the S_r drawn in the S_{r+1} determined by the S_r and the line L (i.e. OL); let the plane (S_2) through L and N meet the S_r in M, then M is the projection of L on the S_r (Art. 10), and

$$(L, S_r) = (L, M);$$

now

$$(L, S'_{n-r}) = \text{comp}\,(L, M)$$

by (7*a*), because M is normal to S'_{n-r} (Art. 9), and because M lies in the S_{n-r+1} determined by S'_{n-r} and L (since it lies in the plane of L and N, and since N lies in S'_{n-r}); hence

$$(L, S_r) = \text{comp}\,(L, S'_{n-r}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7b).$$

14. *Existence of n mutually perpendicular lines at any point of a given S_n.* Let the S_n be defined by n given concurrent lines $OL_1, OL_2, \ldots, OL_n$, no r of which lie in the same S_{r-1}; we wish to draw in the same S_n any set of n lines $OX_1, OX_2, \ldots, OX_n$ which shall be mutually at right angles. Denoting each line through O by a single letter, we first take X_n along L_n, and then by Art. 8 draw, in the given S_n, the following perpendiculars:

X_1 common normal to the $n-1$ lines $L_2L_3L_4\ldots L_{n-1}L_n$,
X_2 " " " " $X_1L_3L_4\ldots L_{n-1}L_n$,
X_3 " " " " $X_1X_2L_4\ldots L_{n-1}L_n$,
X_4 " " " " $X_1X_2X_3\ldots L_{n-1}L_n$,
.....
X_{n-1} " " " " $X_1X_2X_3\ldots X_{n-2}L_n$.

It will then be seen that the n lines $X_1, X_2, \ldots, X_n$ are mutually at right angles. They are sufficient to define the given S_n, since no r of them lie in the same S_{r-1}, because a perpendicular to an S_{r-1} does not lie in the same S_{r-1}, by Art. 6, Cor.

15. *Rectangular axes of co-ordinates.* When the axes are rectangular, the equation (1) takes the more symmetrical form

$$\cos\theta = \cos\alpha_1\cos\beta_1 + \cos\alpha_2\cos\beta_2 + \ldots + \cos\alpha_n\cos\beta_n \quad \ldots\ldots\ldots\ldots(8),$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are the direction angles of the line OV, since we now have

$$x_1 = r\cos\beta_1, \quad x_2 = r\cos\beta_2, \quad \ldots, \quad x_n = r\cos\beta_n \quad \ldots\ldots\ldots\ldots(9),$$

because x_s is now the projection of r on the axis OX_s.

Letting the two directions in (8) coincide, we have

$$\cos^2\beta_1 + \cos^2\beta_2 + \ldots + \cos^2\beta_n = 1 \quad \ldots\ldots\ldots\ldots(10),$$

whence, by (9),

$$r^2 = x_1^2 + x_2^2 + \ldots + x_n^2 \quad \ldots\ldots\ldots\ldots(11).$$

Putting $\theta = 90°$ in equation (8) we have the condition of perpendicularity

$$\cos\alpha_1\cos\beta_1 + \cos\alpha_2\cos\beta_2 + \ldots + \cos\alpha_n\cos\beta_n = 0 \quad \ldots\ldots\ldots\ldots(12).$$

The co-ordinates of any point on the line OV evidently satisfy the $n-1$ equations

$$\frac{x_1}{\cos\beta_1} = \frac{x_2}{\cos\beta_2} = \ldots = \frac{x_n}{\cos\beta_n} \quad \ldots\ldots\ldots\ldots(13);$$

this is called the *symmetrical form* of the $n-1$ equations of a line through O. It follows from (10), (13) that if the equations of a line are given in the form

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n},$$

then its direction cosines will be given by the formula

$$\cos\beta_s = \frac{a_s}{(a_1^2 + a_2^2 + \ldots + a_n^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots(14).$$

16. *Equation of a hyperplane of a given angloid.* Let the angloid be formed by the n rays $L_1, L_2, \ldots, L_n$, which define an S_n, and let their respective direction cosines as to rectangular axes, with the vertex O as origin, be

$$\left.\begin{array}{llll} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots\ldots, & \ldots\ldots, & & \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{array}\right\} \quad \ldots\ldots\ldots\ldots(15);$$

we wish to find the equation of the hyperplane

$$P_r \equiv [L_1 L_2 \ldots (L_r) \ldots L_n];$$

let its equation be

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0 \quad \ldots\ldots\ldots\ldots(16),$$

where $c_1, c_2, \ldots, c_n$ are the direction cosines of its normal; these are to be determined so that this normal shall be perpendicular to each of the lines $L_1, L_2, \ldots, (L_r), \ldots, L_n$; by (12) we have the $n-1$ conditions

$$\begin{aligned} c_1a_{11} + c_2a_{12} + \ldots + c_na_{1n} &= 0, \\ c_1a_{21} + c_2a_{22} + \ldots + c_na_{2n} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_1a_{n1} + c_2a_{n2} + \ldots + c_na_{nn} &= 0, \end{aligned}$$

in which the rth line is of course to be omitted; then, solving by determinants for the $n-1$ ratios, we get

$$\frac{c_1}{A_{r1}} = \frac{c_2}{A_{r2}} = \ldots = \frac{c_n}{A_{rn}};$$

hence the equation of P_r is, by (16),

$$A_{r1}x_1 + A_{r2}x_2 + \ldots + A_{rn}x_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

where A_{rs} is the minor determinant obtained by erasing the rth row and the sth column of the determinant

$$A \equiv \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

and changing signs when $r+s$ is odd.

17. *The face angles of the angloid and of the conjugate angloid.* From (8) and (15) we have

$$\cos(L_1, L_2) = a_{11}a_{21} + a_{12}a_{22} + \ldots + a_{1n}a_{2n},$$

and in general

$$\cos(L_r, L_s) = a_{r1}a_{s1} + a_{r2}a_{s2} + \ldots + a_{rn}a_{sn} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19);$$

this expression is called an invariant function of the a's because it is independent of the particular axes of reference.

Since the conjugate angloid is formed by the normals to the respective hyperplanes of the given angloid, hence the direction cosines of the rays $L_1', L_2', \ldots, L_n'$ are, by Art. 16, respectively proportional to

$$\left.\begin{matrix} A_{11}, & A_{12}, & \ldots, & A_{1n} \\ A_{21}, & A_{22}, & \ldots, & A_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ A_{n1}, & A_{n2}, & \ldots, & A_{nn} \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20);$$

hence, by (8) and (14), we see that the cosines of the face angles of the conjugate angloid are given by the formula

$$\cos(L_r', L_s') = \frac{A_{r1}A_{s1} + A_{r2}A_{s2} + \ldots + A_{rn}A_{sn}}{(A_{r1}^2 + A_{r2}^2 + \ldots + A_{rn}^2)^{\frac{1}{2}}\,(A_{s1}^2 + A_{s2}^2 + \ldots + A_{sn}^2)^{\frac{1}{2}}} \ldots (21).$$

We shall see in Art. 19 that the numerator and the factors of the denominator are invariant functions; and we shall show how to express them in forms of simpler invariants, which involve directly the face angles of the given angloid.

18. *The determinant A expressed as an invariant in terms of the face angles of the angloid.* Squaring this determinant by the usual rule, we have

$$A^2 = \begin{vmatrix} a_{11}^2 + a_{12}^2 + \ldots, & a_{11}a_{21} + a_{12}a_{22} + \ldots, & a_{11}a_{31} + a_{12}a_{32} + \ldots, & \ldots \\ a_{11}a_{21} + a_{12}a_{22} + \ldots, & a_{21}^2 + a_{22}^2 + \ldots, & a_{21}a_{31} + a_{22}a_{32} + \ldots, & \ldots \\ a_{11}a_{31} + a_{12}a_{32} + \ldots, & a_{21}a_{31} + a_{22}a_{32} + \ldots, & a_{31}^2 + a_{32}^2 + \ldots, & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{11}a_{n1} + a_{12}a_{n2} + \ldots, & a_{21}a_{n1} + a_{22}a_{n2} + \ldots, & a_{31}a_{n1} + a_{32}a_{n2} + \ldots, & \ldots \end{vmatrix};$$

hence by (8) and (19)

$$A^2 = \begin{vmatrix} 1, & \cos(1,2), & \cos(1,3), & \ldots, & \cos(1,n) \\ \cos(1,2), & 1, & \cos(2,3), & \ldots, & \cos(2,n) \\ \cos(1,3), & \cos(2,3), & 1, & \ldots, & \cos(3,n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \cos(1,n), & \cos(2,n), & \cos(3,n), & \ldots, & 1 \end{vmatrix} \equiv R, \text{ say} \ldots (22),$$

where the plane angle (L_1, L_2) is written (1, 2), etc., and where R stands for the symmetrical determinant whose elements are the cosines of the face angles (1, 2), (1, 3), ..., the main diagonal consisting of units.

19. *Some symmetric functions of the minors of A expressed as invariants in terms of the minors of R.* Squaring the reciprocal determinant of A (that is, the determinant formed by its minors), we have

$$\begin{vmatrix} A_{11}, & A_{12}, & \ldots, & A_{1n} \\ A_{21}, & A_{22}, & \ldots, & A_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_{n1}, & A_{n2}, & \ldots, & A_{nn} \end{vmatrix}^2.$$

$$= \begin{vmatrix} A_{11}^2 + A_{12}^2 + \ldots, & A_{11}A_{21} + A_{12}A_{22} + \ldots, & \ldots, & A_{11}A_{n1} + A_{12}A_{n2} + \ldots \\ A_{11}A_{21} + A_{12}A_{22} + \ldots, & A_{21}^2 + A_{22}^2 + \ldots, & \ldots, & A_{21}A_{n1} + A_{22}A_{n2} + \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_{11}A_{n1} + A_{12}A_{n2} + \ldots, & A_{21}A_{n1} + A_{22}A_{n2} + \ldots, & \ldots, & A_{n1}^2 + A_{n2}^2 + \ldots \end{vmatrix}.$$

Now by the theory of determinants, the reciprocal determinant of A is identical with A^{n-1}, hence the left-hand member is $A^{2(n-1)}$, that is R^{n-1}, which is again identical with the reciprocal determinant of R, namely

$$\begin{vmatrix} R_{11}, & R_{12}, & \ldots, & R_{1n} \\ R_{12}, & R_{22}, & \ldots, & R_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ R_{1n}, & R_{2n}, & \ldots, & R_{nn} \end{vmatrix},$$

hence this determinant is identical with the determinant on the right above; equating corresponding terms we have a series of identities which may all be included in the following general formula:

$$R_{rs} = A_{r1}A_{s1} + A_{r2}A_{s2} + \ldots + A_{rn}A_{sn} \quad \ldots\ldots\ldots\ldots\ldots (23),$$

the letters r and s each denoting any integer from 1 to n. This set of invariant functions of the A's enters into various fundamental expressions relating to the n-dimensioned angle.

By the aid of (23) we can now write (21) in the simpler form

$$\cos(L_r', L_s') = \frac{R_{rs}}{(R_{rr}R_{ss})^{\frac{1}{2}}} \quad \text{.....(24);}$$

we have also

$$\sin^2(L_r', L_s') = 1 - \cos^2(L_r', L_s') = \frac{R_{rr}R_{ss} - R_{rs}^2}{R_{rr}R_{ss}},$$

in which the numerator is a determinant of the minors of R, and hence

$$\sin^2(L_r', L_s') = \frac{RR_{rs,rs}}{R_{rr}R_{ss}} \quad \text{.....(24}a\text{),}$$

where $R_{rs,rs}$ is the second minor obtained by omitting the rth and sth rows and the rth and sth columns of R.

20. *Dihedrals of angloid in terms of face angles.* Since the plane angle of the dihedral (P_r, P_s) equals the supplement of the angle between the normals (L_r', L_s'), by (4), hence

$$\left.\begin{aligned} \cos(P_r, P_s) &= \frac{-R_{rs}}{(R_{rr}R_{ss})^{\frac{1}{2}}} \\ \sin(P_r, P_s) &= \left(\frac{RR_{rs,rs}}{R_{rr}R_{ss}}\right)^{\frac{1}{2}} \end{aligned}\right\} \quad \text{.....(25).}$$

Both of these forms are useful in various connections; they are generalizations of the familiar fundamental formulae in spherical trigonometry, which may be written analogously for the three-dimensioned angle (L_1, L_2, L_3) in the form

$$\left.\begin{aligned} \cos(P_1, P_2) &= \frac{\begin{vmatrix} \cos(1,2), & \cos(2,3) \\ \cos(1,3), & 1 \end{vmatrix}}{\begin{vmatrix} 1, & \cos(2,3) \\ \cos(2,3), & 1 \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} 1, & \cos(1,3) \\ \cos(1,3), & 1 \end{vmatrix}^{\frac{1}{2}}} \\ &= \frac{\cos(1,2) - \cos(1,3)\cos(2,3)}{\sin(1,3)\sin(2,3)}, \\ \sin(P_1, P_2) &= \frac{R^{\frac{1}{2}}}{\sin(1,3)\sin(2,3)} \end{aligned}\right\} \quad \text{...(26).}$$

21. *Face angles in terms of dihedrals.* We obtain the "polar" formula of (25) by applying the latter to the conjugate angloid

$$\alpha' \equiv (L_1', L_2', \ldots, L_n'),$$

and using the supplementary relations (4, 5), between the two angloids, and changing signs, then

$$\left.\begin{aligned} \cos(L_r, L_s) &= \frac{\mathfrak{R}_{rs}}{(\mathfrak{R}_{rr}\mathfrak{R}_{ss})^{\frac{1}{2}}} \\ \sin(L_r, L_s) &= \left(\frac{\mathfrak{R}\mathfrak{R}_{rs,rs}}{\mathfrak{R}_{rr}\mathfrak{R}_{ss}}\right)^{\frac{1}{2}} \end{aligned}\right\} \quad \text{.....(27),}$$

in which the minors belong to the determinant $\mathfrak{R}$, where

$$\mathfrak{R} \equiv \begin{vmatrix} 1, & -\cos(P_1, P_2), & -\cos(P_1, P_3), & \ldots, & -\cos(P_1, P_n) \\ -\cos(P_1, P_2), & 1, & -\cos(P_2, P_3), & \ldots, & -\cos(P_2, P_n) \\ -\cos(P_1, P_3), & -\cos(P_2, P_3), & 1, & \ldots, & -\cos(P_3, P_n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ -\cos(P_1, P_n), & -\cos(P_2, P_n), & -\cos(P_3, P_n), & \ldots, & 1, \end{vmatrix} \quad \ldots(28).$$

As a particular case, putting $n = 3$, $r = 1$, $s = 2$, we have the familiar polar formulae of (26):

$$\left.\begin{aligned} \cos(L_1, L_2) &= \frac{\cos(P_1, P_2) + \cos(P_1, P_3)\cos(P_2, P_3)}{\sin(P_1, P_3)\sin(P_2, P_3)} \\ \sin(L_1, L_2) &= \frac{\mathfrak{R}}{\sin(P_1, P_3)\sin(P_2, P_3)} \end{aligned}\right\} \quad \ldots\ldots\ldots(29),$$

giving the cosine and sine of a face angle of a trihedral in terms of the dihedrals.

22. *Dihedrals of various orders.* In the case of the $(n-1)$-dimensioned angloid

$$\alpha_r \equiv [L_1, L_2, \ldots (L_r) \ldots, L_n],$$

whose hyperplanes are $P_{r1}, P_{r2}, \ldots$, Art. 11, it is sometimes necessary to express the cosine of a dihedral angle (P_{rs}, P_{rt}) in terms of dihedrals of the original angloid α; and we shall show that it can be expressed in terms of the three dihedrals (P_r, P_s), (P_r, P_t), (P_s, P_t).

We take the three-dimensioned angle (L_r', L_s', L_t') and write down from (25) the cosine of the dihedral between the planes $(L_r'L_s')$ and $(L_r'L_t')$:

$$\cos(L_r'L_s', L_r'L_t') = \frac{\cos(s', t') - \cos(r', s')\cos(r', t')}{\sin(r', s')\sin(r', t')};$$

we then convert this into its polar formula, using the supplemental relations (4, 5), and obtain

$$\cos(P_{rs}, P_{rt}) = \frac{\cos(P_s, P_t) + \cos(P_r, P_s)\cos(P_r, P_t)}{\sin(P_r, P_s)\sin(P_r, P_t)} \quad \ldots\ldots\ldots(30).$$

Again in the case of the $(n-2)$-angloid

$$\alpha_{rs} \equiv [L_1, L_2, \ldots (L_r L_s), \ldots, L_n],$$

whose hyperplanes are $P_{rs1}, P_{rs2}, \ldots$, we wish to express the cosine of a dihedral (P_{rst}, P_{rsu}) in terms of the original dihedrals (P_r, P_s), (P_r, P_t), (P_r, P_u), (P_s, P_t), (P_s, P_u), (P_t, P_u).

We take the four-angloid (L_r', L_s', L_t', L_u') and write the expression for $\cos(L_r'L_s'L_t', L_r'L_s'L_u')$; we then convert it into its polar formula, using the supplemental relations like (4, 5, 6), and obtain

$$\cos(P_{rst}, P_{rsu}) = \frac{\mathfrak{R}^{iv}_{tu}}{(\mathfrak{R}^{iv}_{tt}\,\mathfrak{R}^{iv}_{uu})^{\frac{1}{2}}} \quad \ldots\ldots\ldots(31)$$

in which the determinants on the right are minors of the fourth order determinant

$$\mathfrak{R}^{\text{iv}} = \begin{vmatrix} 1, & -\cos(P_r, P_s), & -\cos(P_r, P_t), & -\cos(P_r, P_u) \\ -\cos(P_r, P_s), & 1, & -\cos(P_s, P_t), & -\cos(P_s, P_u) \\ -\cos(P_r, P_t), & -\cos(P_s, P_t), & 1, & -\cos(P_t, P_u) \\ -\cos(P_r, P_u), & -\cos(P_s, P_u), & -\cos(P_t, P_u), & 1 \end{vmatrix} \quad \ldots(32)$$

which is formed from the pth, qth, sth, and tth rows of the determinant $\mathfrak{R}$ and from the pth, qth, sth, and tth columns of the same determinant; the minor $\mathfrak{R}^{\text{iv}}_{st}$ being formed from $\mathfrak{R}^{\text{iv}}$ by erasing the row in which s is a common subscript, and the column in which t is a common subscript, and changing sign if the sum of the order numbers of the row and column in $\mathfrak{R}^{\text{iv}}$ is odd.

Similarly, in the case of the $(n-3)$-dimensioned angle α_{rst} whose hyperplanes are P_{rstu}, P_{rstv}, ..., we have

$$\cos(P_{rstu}, P_{rstv}) = \frac{\mathfrak{R}^{\text{v}}_{uv}}{(\mathfrak{R}^{\text{v}}_{uu}\mathfrak{R}^{\text{v}}_{vv})^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(33),$$

where these determinants are minors of the fifth-order determinant $\mathfrak{R}^{\text{v}}$ which is formed from the rth, sth, tth, uth, and vth rows and columns of the main determinant $\mathfrak{R}$. This rule will evidently apply to dihedrals of any order.

It is also sometimes important to reverse this process, and to express the cosines of the original dihedrals in terms of those of lower orders. If we take the three-dimensioned angle (L_r', L_s', L_t') and write down from (27) the cosine of the face angle (L_r', L_s') in terms of dihedrals we get

$$\cos(L_r', L_s') = \frac{\cos(L_r'L_t', L_s'L_t') + \cos(L_r'L_s', L_s'L_t')\cos(L_r'L_s', L_r'L_t')}{\sin(L_r'L_s', L_s'L_t')\sin(L_r'L_s', L_r'L_t')};$$

if we then take the polar formula of this using the supplemental relations in the original angloid (4, 5, 6) we obtain

$$\cos(P_r, P_s) = \frac{\cos(P_{rt}, P_{st}) - \cos(P_{rs}, P_{rt})\cos(P_{rs}, P_{st})}{\sin(P_{rs}, P_{rt})\sin(P_{rs}, P_{st})} \quad \ldots\ldots(34).$$

Similarly to express $\cos(P_r, P_s)$ in terms of dihedrals of $(n-2)$-angloids we take the four-angloid (L_r', L_s', L_t', L_u') and write down the cosine of the face angle (L_r', L_s') in terms of dihedrals such as $(L_r'L_s'L_t', L_r'L_s'L_u')$; we then convert this into its polar formula, using the supplemental relations in the original angloid like (4, 5, 6), and obtain

$$\cos(P_r, P_s) = \frac{S_{rs}}{(S_{rr}S_{ss})^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(35),$$

in which the determinants are minors of the determinant

$$S \equiv \begin{vmatrix} 1, & \cos(P_r, P_s)_{tu}, & \cos(P_r, P_t)_{su}, & \cos(P_r, P_u)_{st} \\ \cos(P_r, P_s)_{tu}, & 1, & \cos(P_s, P_t)_{ru}, & \cos(P_s, P_u)_{rt} \\ \cos(P_r, P_t)_{su}, & \cos(P_s, P_t)_{ru}, & 1, & \cos(P_t, P_u)_{rs} \\ \cos(P_r, P_u)_{st}, & \cos(P_s, P_u)_{rt}, & \cos(P_t, P_u)_{rs}, & 1 \end{vmatrix} \quad \ldots(36),$$

where $(P_r, P_s)_{tu}$ is an abbreviation for (P_{rtu}, P_{stu}), etc.; the minor S_{rs} being obtained by erasing the row in which P_r is common and the column in which P_s is common.

The generality of (35) is obvious; and as the order of the angloid further diminishes the number of subscripts in (36) increases, and so does the order of the determinant S, but the same rule of formation applies. Equations like (33) and (35) will be used in Part II to furnish the relations among correlation coefficients of various orders mentioned in the introductory quotation from Professor Karl Pearson.

23. *Inclination of a ray to the corresponding hyperplane; and to any planoid.* Since we have from (7)

$$(L_r, P_r) = \text{comp}\,(L_r, L_r'),$$
$$\sin(L_r, P_r) = \cos(L_r, L_r'),$$

hence, from (8), (14), (15), (20),

$$\sin(L_r, P_r) = \frac{a_{r1}A_{r1} + a_{r2}A_{r2} + \ldots + a_{rn}A_{rn}}{(a_{r1}^2 + a_{r2}^2 + \ldots + a_{rn}^2)^{\frac{1}{2}}(A_{r1}^2 + A_{r2}^2 + \ldots + A_{rn}^2)^{\frac{1}{2}}},$$

but the numerator equals the determinant A because A_{rs} is the minor corresponding to the element a_{rs}, hence, from (10), (22), (23),

$$\sin(L_r, P_r) = \left(\frac{R}{R_{rr}}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

It also follows that the inclination of the ray L_r to the planoid P_{rs} is given by

$$\sin(L_r, P_{rs}) = \left(\frac{R_{ss}}{R_{rs,\,rs}}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38),$$

because when the ray L_s is omitted the determinant R reduces to its first minor R_{ss}, and the first minor R_{rr} reduces to the second minor $R_{rs,\,rs}$.

Similarly the inclination of L_r to the planoid P_{rst} is obtained from

$$\sin(L_r, P_{rst}) = \left(\frac{R_{st,\,st}}{R_{rst,\,rst}}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39)$$

and so on.

For reasons of analogy which appear more clearly later, the square root of the determinant R is called the *sine* of the angloid

$$\alpha \equiv (L_1, L_2, \ldots, L_n);$$

thus we may write

$$R^{\frac{1}{2}} = \sin\alpha = \sin(1, 2, \ldots n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40),$$

$$R^{\frac{1}{2}}_{rr} = \sin\alpha_r = \sin[1, 2, \ldots (r) \ldots, n] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41),$$

$$R^{\frac{1}{2}}_{rs,\,rs} = \sin\alpha_{rs} = \sin[1, 2, \ldots (rs) \ldots, n] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42),$$

hence (37) may be put in the form

$$\sin[r, 12 \ldots (r) \ldots n] = \frac{\sin(1, 2, \ldots, n)}{\sin[1, 2, \ldots (r) \ldots, n]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43),$$

$$\sin[r, 12 \ldots (rs) \ldots n] = \frac{\sin[1, 2, \ldots (s) \ldots, n]}{\sin[1, 2, \ldots (rs) \ldots, n]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44);$$

in particular

$$\sin(3, 12) = \frac{\sin(1, 2, 3)}{\sin(1, 2)} \quad \text{.....(45)}$$

$$\sin(4, 123) = \frac{\sin(1, 2, 3, 4)}{\sin(1, 2, 3)} \quad \text{..... .....(46).}$$

Hypersphere; hyperspherical polygon as trace of angloid; arcs as measures of angles. In any S_n the locus of a point whose distance from a given point O is equal to a given constant r is called a *hypersphere* whose centre is O and whose radius is r; its equation, from (11), is

$$x_1^2 + x_2^2 + \ldots + x_n^2 = r^2 \quad \text{.....(47).}$$

The intersection of the hypersphere with a hyperplane passing through O is a *hyper-great-circle.* The rays $L_1, L_2, \ldots, L_n$, of an n-dimensioned angle whose vertex is at O, meet the hypersphere in n points $1, 2, \ldots, n$, which are the vertices of a *hyperspherical polygon* whose sides are n hyper-great-circles on each of which $n-1$ of the vertices lie; and through each vertex $n-1$ of the sides pass. The rays $L_1', L_2', \ldots L_n'$ of the conjugate angloid meet the hypersphere in n points $1', 2', \ldots, n'$ which are the *poles* of the respective sides, the ray L_r' being normal to P_r (Art. 11) and situated at the same side of P_r as L_r is.

The face angle (L_r, L_s) is measured by the arc (r, s) and the dihedral (P_r, P_s) is measured by the supplement of the arc (r', s'); and the angle (L_r, P_r) is evidently, from Art. 13, measured by the arc p_r drawn from the vertex r perpendicular to the opposite side P_r, that is by the complement of the arc (r, r'), thus by (37)

$$\sin p_r = \sin(r, P_r) = \cos(r, r') = \left(\frac{R}{R_{rr}}\right)^{\frac{1}{2}} \quad \text{.....(48).}$$

Relation between p_r and $p_{r(s)}$. In the applications it is useful to have a relation between the perpendicular arc p_r and the corresponding perpendicular $p_{r(s)}$ drawn from the same point (r) to the opposite side in the partial polygon obtained by omitting the ray L_s. We have from (38)

$$\sin p_{r(s)} = \sin(r, P_{rs}) = \left(\frac{R_{ss}}{R_{rs,\,rs}}\right)^{\frac{1}{2}} \quad \text{.....(49);}$$

then dividing (48) by (49) and using (24 a) we get

$$\frac{\sin p_r}{\sin p_{r(s)}} = \sin(r', s') \quad \text{.....(50),}$$

the required relation.

It may be of interest to verify this relation by ordinary spherical trigonometry: writing it in the form

$$\sin(r', s') = \frac{\sin p_r}{\sin p_{r(s)}} = \frac{\sin(r, P_r)}{\sin(r, P_{rs})},$$

and using (7) and (7 b), we have

$$\sin(r', s') = \frac{\cos(r, r')}{\cos(r, r's')} \quad \text{.....(51),}$$

which is a simple property of the spherical triangle (r, r', s') having the side rs' a quadrant, where $(r, r's')$ means the perpendicular arc from the vertex r to the

opposite side $r's'$; this furnishes the required verification; and could, if desired, be used as a starting point to prove a number of the preceding formulae by reversing the order of derivation.

24. *Volume of n-dimensioned parallelopiped.* Let one corner be at the origin O, and let the n adjacent edges be the vectors $OA_1, OA_2, \ldots, OA_n$, whose lengths are $a_1, a_2, \ldots, a_n$, and whose direction cosines with regard to n given rectangular axes are respectively:

$$\left.\begin{array}{l} a_{11}, a_{12}, \ldots, a_{1n} \\ a_{21}, a_{22}, \ldots, a_{2n} \\ \ldots\ldots\ldots\ldots \\ a_{n1}, a_{n2}, \ldots, a_{nn} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52);$$

then, following the successive steps in Art. 4, we see that the area of the parallelogram generated by translating OA_1 through a distance represented in magnitude and direction by the vector OA_2 is measured by the product of OA_1 by the component of OA_2 in the direction normal to OA_1, therefore

$$\text{pgm } OA_1A_2 = a_1a_2 \sin (1, 2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (53);$$

again, the volume generated by translating the parallelogram OA_1A_2 through a distance represented in magnitude and direction by the vector OA_3 is measured by the product of OA_1A_2 by the component of OA_3 normal to the plane OA_1A_2, therefore

$$\text{ppd } OA_1A_2A_3 = OA_1A_2 \times a_3 \sin (3, 12),$$

hence by (45) and (53)

$$\text{ppd } OA_1A_2A_3 = a_1a_2a_3 \sin (1, 2, 3) \ldots\ldots\ldots\ldots\ldots\ldots (54);$$

next, the volume of the four-dimensioned parallelopiped, generated by translating the three-dimensioned one $OA_1A_2A_3$ in the direction and amount indicated by the vector OA_4, is defined by analogy to have the measure represented by the product of $OA_1A_2A_3$ by the component of OA_4 normal to the hyperplane of the rays (1, 2, 3) situated in the four-space of the rays (1, 2, 3, 4), therefore

$$\text{ppd } OA_1A_2A_3A_4 = OA_1A_2A_3 \times a_4 \sin (4, 123),$$

hence, by (46) and (54),

$$\text{ppd } OA_1A_2A_3A_4 = a_1a_2a_3a_4 \sin (1, 2, 3, 4) \ldots\ldots\ldots\ldots\ldots (55);$$

proceeding thus, we find in general

$$\text{ppd } OA_1A_2 \ldots A_n = a_1a_2 \ldots a_n \sin (1, 2, \ldots, n) \ldots\ldots\ldots\ldots\ldots (56);$$

another useful form for this volume is obtained by replacing $\sin (1, 2, \ldots, n)$, which is the square root of the determinant R, by the determinant A formed by the direction cosines of the n vectors [see equation (22)]; and from this we easily derive another expression involving the co-ordinates of the extremities $A_1, A_2, \ldots, A_n$ of these vectors, viz.:

$$\text{ppd } OA_1A_2 \ldots A_n = \begin{vmatrix} x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)} \\ x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ x_1^{(n)}, & x_2^{(n)}, & \ldots, & x_n^{(n)} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots (57),$$

in which the co-ordinates of the point A_p are denoted by

$$x_1^{(p)}, x_2^{(p)}, \ldots, x_n^{(p)};$$

the identity of these two determinant forms will be seen from the fact that

$$x_s^{(p)} = a_p \cos(OA_p, OX_s) = a_p a_{ps}.$$

25. *Linear correspondence between two n-spaces.* Let the co-ordinates of a point P in a space S, referred to rectangular axes $OX_1, OX_2, \ldots, OX_n$ in that space, be $(x_1, x_2, \ldots, x_n)$, and let the co-ordinates of a point P' in a space S' referred to rectangular axes $O'X_1', O'X_2', \ldots, O'X_n'$ in that space be $(x_1', x_2', \ldots x_n')$, and let the two sets of variable co-ordinates be connected by fixed linear relations of the form

$$\left.\begin{aligned} x_1 &= a_{11}x_1' + a_{12}x_2' + \ldots + a_{1n}x_n' \\ x_2 &= a_{21}x_1' + a_{22}x_2' + \ldots + a_{2n}x_n' \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= a_{n1}x_1' + a_{n2}x_2' + \ldots + a_{nn}x_n' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (58),$$

then the points P and P' will be called *corresponding points* in the two spaces, and when the position of one of them is given, the position of the other is completely determined by means of the given relations; when one of them describes a hyperplane in its space, the other describes a corresponding hyperplane in the other space, for, suppose that the co-ordinates of P satisfy the equation

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = k,$$

then substituting from (58) we have

$$c_1(a_{11}x_1' + a_{12}x_2' + \ldots) + c_2(a_{21}x_1' + a_{22}x_2' + \ldots) + \ldots = k,$$

an equation which must be satisfied by the co-ordinates of P', and represents a hyperplane in the space S'. It is evident that if two hyperplanes are parallel in S, then their corresponding hyperplanes are parallel in S', for the equations of two parallel hyperplanes differ only in the constant terms, hence the two transformed equations will differ only in the two constant terms, and will thus represent two parallel hyperplanes. It follows easily that if two lines are parallel in S, then their two corresponding lines are parallel in S', for if one of the lines in S is represented by $n-1$ linear equations, then the parallel line may be represented by $n-1$ equations differing respectively from the former set in only the constant terms, hence the respective transformed equations will differ only in the constant terms, and will thus represent two parallel lines in S'. We conclude that if either point describes an n-dimensioned parallelopiped, then the other point describes a corresponding one; when one of these is rectangular the other is in general not rectangular, for instance, the hyperplanes $x_1 = 0$ and $x_2 = 0$, which are at right angles, correspond to the hyperplanes $a_{11}x_1' + a_{12}x_2' + \ldots = 0$ and $a_{21}x_1' + a_{22}x_2' + \ldots = 0$, and these are not at right angles unless $a_{11}a_{21} + a_{12}a_{22} + \ldots = 0$, which is generally not satisfied.

We shall now prove the useful principle that the volumes of any two corresponding regions of the two spaces have a constant ratio, and we shall express

this constant in terms of the coefficients of the given linear transformation. We first consider a small rectangular parallelopiped having one corner at $(x_1', x_2', \ldots, x_n')$ and the opposite corner at $(x_1' + dx_1', x_2' + dx_2', \ldots, x_n' + dx_n')$, then the corresponding parallelopiped will have one corner at $(x_1, x_2, \ldots, x_n)$, and the relative co-ordinates of the other extremities of the n adjacent sides are, respectively, by differentiating (58),

$$\left.\begin{array}{l} a_{11}\,dx_1',\ a_{21}\,dx_1',\ \ldots,\ a_{n1}\,dx_1' \\ a_{12}\,dx_2',\ a_{22}\,dx_2',\ \ldots,\ a_{n2}\,dx_2' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}dx_n',\ a_{2n}dx_n',\ \ldots,\ a_{nn}dx_n' \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(59),$$

hence, by (57), its volume is

$$dV = \begin{vmatrix} a_{11}, & a_{21}, \ldots, & a_{n1} \\ a_{12}, & a_{22}, \ldots, & a_{n2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}, & a_{2n}, \ldots, & a_{nn} \end{vmatrix} dx_1'dx_2' \ldots dx_n' \ldots\ldots\ldots\ldots(60)$$

i.e. $$dV = A\,dV' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(61),$$

where A is the determinant of the given linear transformation; we can now integrate this differential equation over any two corresponding regions of S and S', and obtain

$$V = A\,V' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62).$$

PART II.

Application to Correlation Theory for n Variables.

26. *Normal Frequency Distribution.* We begin with the typical case, in which the correlation is normal, and there is thus a frequency-function having the exponential type*

$$f = f_0 e^{-\frac{1}{2}Q} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

where Q is a general homogeneous quadratic function of the n deviations, and may be conveniently written in the form

$$Q \equiv c_{11}\left(\frac{x_1}{\sigma_1}\right)^2 + c_{22}\left(\frac{x_2}{\sigma_2}\right)^2 + \ldots + 2c_{12}\left(\frac{x_1x_2}{\sigma_1\sigma_2}\right) + \ldots \ldots\ldots\ldots\ldots(64).$$

In this x_i is the deviation of the ith characteristic in any individual from its average value† taken with regard to the group of N individuals whose n character-

* This is a natural generalization of the frequency-function for n independent characteristics, which is the product of n simple Gaussian functions of the normal form $e^{-x^2/(2\sigma^2)}$, a case in which Q has only the square terms; the product terms in (64) may be regarded as arising from a linear transformation to a new set of variables as explained in Art. 25. The only restriction on Q is that it must be positive for all real values of the variables. This requires the coefficients of the type C_{ii} to be positive, and the discriminants of the type $C_{ii}C_{jj} - C_{ij}^2$ to be also positive.

† In the few cases in which it is necessary to indicate the pth characteristic itself, we denote it by X_p and its mean by $\overline{X}_p$, thus $x_p = X_p - \overline{X}_p$.

istics we are comparing, and σ_i is the standard deviation of the ith characteristic, that is to say, σ_i^2 is the average of the N values of x_i^2, thus

$$\sigma_i^2 = \frac{\Sigma x_i^2}{N} \quad \text{.....(65),}$$

in which as elsewhere throughout this essay, the symbol Σ means a summation with regard to the N individuals; in what follows we regard each deviation as expressed by its ratio to the corresponding standard deviation, so that what is above written as x/σ will be denoted by x_i, thus

$$x_i \equiv \frac{x_i}{\sigma_i} \quad \text{.....(66);}$$

it will be found that this notation simplifies both the algebraic work and the geometrical interpretation, and has the advantage of making all of our fundamental variables pure numbers (or ratios), while some of the original variables may be weights, others lengths, others grades of intelligence, others degrees of temperature, etc.

The frequency-function f mentioned above measures what may be called the "density" of the frequency distribution in the immediate vicinity of the values of the characteristic deviations represented by $x_1, x_2, \ldots, x_n$; this means that the product

$$f dx_1 dx_2 \ldots dx_n$$

represents the number of individuals dF whose n deviations lie respectively between x_1 and $x_1 + dx_1$, between x_2 and $x_2 + dx_2$, ..., and between x_n and $x_n + dx_n$, so that we may write

$$dF = f dx_1 dx_2 \ldots dx_n;$$

this use of the word "density" will be clearer if we imagine an n-dimensioned correlation chart* ruled up into little "cells" by lines drawn parallel to n mutually rectangular axes passing through the centre of the chart taken as origin, and if we suppose an individual whose n deviations lie respectively between x_1 and $x_1 + dx_1$, between x_2 and $x_2 + dx_2$, etc., to be represented by a dot placed in the cell which has two of its opposite corners at the points whose co-ordinates are $x_1, x_2, \ldots, x_n$ and $x_1 + dx_1, x_2 + dx_2, \ldots, x_n + dx_n$, and similarly for all other cells†, then, if we let dF represent the number of dots in the cell just mentioned and take

$$dF = f dx_1 dx_2 \ldots dx_n \quad \text{.....(67),}$$

we can say that f is the "density" of the distribution of dots in the vicinity of the point $(x_1, x_2, \ldots, x_n)$ estimated per unit of volume, and the frequency F attached to any assigned region of the chart is to be obtained from (67) by a multiple integration over that region; thus the important distinction between the frequency F (attached to an assigned region of the space) and the frequency-function f (at an

* We assume an elementary knowledge of the conception of a frequency chart of two dimensions.

† An individual placed at the "centre" of the chart possesses the average value of each of the n separate characteristics, and thus possesses zero values of each of the n deviations.

assigned point) is closely analogous to the familiar distinction in physics between the mass (attached to an assigned volume) and the mass-function (or density at an assigned point).

The constant f_0 is the maximum (or modal) value of f, and corresponds to the zero deviations $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$. The locus of points where there is the same given value of f is $f = k$, whose equation is

$$Q \equiv c_{11}x_1^2 + c_{22}x_2^2 + \ldots + 2c_{12}x_1x_2 + \ldots = k' \quad \ldots\ldots\ldots\ldots(68),$$

in which k' depends on the assigned value of f. When f takes on a succession of values, so does k', and this equation represents a family of equal-frequency surfaces, in which k is the parameter of the family.

When there are no product terms in Q there is no correlation among the variable characteristics, because f then breaks up into the product of n separate and independent Gaussian frequency-functions. This suggests the transformation in the next Article.

27. *Transformation to a Set of New and Uncorrelated Variables.* We suppose Q resolved into the sum of the squares of n linear functions of $x_1, x_2, \ldots, x_n$, in the form*

$$Q \equiv x_1'^2 + x_2'^2 + \ldots + x_n'^2 \quad \ldots\ldots\ldots\ldots(69);$$

the new variables may be regarded as the deviations of a set of uncorrelated measurements, since they have independent frequency distributions as indicated above. In terms of these new and uncorrelated variables, our original variables have linear expressions of the form

$$\left.\begin{aligned} x_1 &= a_{11}x_1' + a_{12}x_2' + \ldots + a_{1n}x_n' \\ x_2 &= a_{21}x_1' + a_{22}x_2' + \ldots + a_{2n}x_n' \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= a_{n1}x_1' + a_{n2}x_2' + \ldots + a_{nn}x_n' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(70);$$

in which the constant coefficients are of course the same for each member of the group of N individuals whose n characteristics we are comparing. The uncorrelated characteristics may be regarded as having independent Gaussian distributions among the N individuals, and as representing hidden elemental capacities on which the outward manifestations functionally depend; and from this functionality the observed correlations may be considered to arise. When the functionality between the two sets of deviations is linear as above the resulting correlation is normal. Whatever be the form of the functional dependence among the measurements the deviations will have approximately linear relations like (70) when the higher powers of these deviations are relatively small; this may serve to explain the fact that so many of the usual correlations are practically normal within certain limits.

* It is known in the theory of quantics that this resolution can always be performed by a transformation with real coefficients, and that the new terms are all positive, when Q is essentially positive. See first footnote to Art. 26, p. 192. The *possibility* of this resolution is all that concerns us; there are many possible ways of doing it. One mode of resolution which has a special interest is given in Art. 32.

28. *Elementary summations.* Letting the symbol Σ stand for summation with regard to the N individuals we have

$$\Sigma x_1^2 = \Sigma \left(\frac{x_1}{\sigma_1}\right)^2 = \frac{\Sigma x_1^2}{\sigma_1^2} = \frac{N\sigma_1^2}{\sigma_1^2} = N;$$

similarly
$$\Sigma x_p^2 = N, \quad \Sigma x_p'^2 = N \quad \text{.....(71);}$$

again
$$\Sigma x_1 x_2 = \frac{\Sigma x_1 x_2}{\sigma_1 \sigma_2} = \frac{\Sigma x_1 x_2}{N\sigma_1\sigma_2} N = N r_{12} \quad \text{.....(72),}$$

where r_{12} is the coefficient of correlation* between the two variables x_1 and x_2; similarly

$$\Sigma x_p x_q = N r_{pq}; \quad \Sigma x_p' x_q' = 0 \quad \text{.....(73),}$$

since the variables x_p' and x_q' are uncorrelated.

Squaring each of equations (70) and performing a summation on each equation with regard to the N individuals we have from (71), after dividing by N,

$$\left.\begin{aligned} 1 &= a_{11}^2 + a_{12}^2 + \ldots + a_{1n}^2 \\ 1 &= a_{21}^2 + a_{22}^2 + \ldots + a_{2n}^2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 &= a_{n1}^2 + a_{n2}^2 + \ldots + a_{nn}^2 \end{aligned}\right\} \quad \text{.....(74);}$$

again multiplying the expressions for x_1 and x_2 and averaging over the N individuals we have, from (72),

$$r_{12} = a_{11}a_{21} + a_{12}a_{22} + \ldots + a_{1n}a_{2n};$$

similarly
$$r_{pq} = a_{p1}a_{q1} + a_{p2}a_{q2} + \ldots + a_{pn}a_{qn} \quad \text{.....(75).}$$

The form of equation (75) suggests a geometric interpretation in n-dimensioned space.

29. *Transformed frequency chart.* Corresponding to the frequency chart C on which the n deviations $x_1, x_2, \ldots, x_n$ are represented by the co-ordinates of a point P as explained in Art. 26, we construct a frequency chart C' on which the n deviations $x_1', x_2', \ldots, x_n'$ are similarly represented by the co-ordinates of a point P' with regard to rectangular axes $O'X_1'$, $O'X_2'$, ..., $O'X_n'$, through the centre O' of the chart C'. The correspondence between P and P' is defined by means of the linear relations (70) and is of the kind considered in Art. 25†; when one of these points is made to move over an assigned region in its space, the other moves over the corresponding region in the other space; and, since the set of deviations represented by P and the set represented by P' belong to the same individuals of the group, it is clear that the numbers of individuals or "dots" attached to two corresponding regions are identical; thus

$$f dV = f' dV' \quad \text{.....(76),}$$

* We assume an elementary knowledge of correlation theory for two variables. The division of the deviations x_i by σ_i is for purpose of standardizing the scales.

† It will appear in the sequel that many of our operations are much simplified when transformed to the space C'; hence the importance of a clear conception of the correspondence between the two spaces.

where f is the number of dots per unit volume in the vicinity of P, and f' is the number of dots per unit volume, in the vicinity of P'; but, as in Art. 25

$$dV = A\,dV' \quad \text{.....(77),}$$

where A is the determinant of the transformation, therefore

$$f' = Af \quad \text{.....(78);}$$

thus the values of the frequency-function at corresponding points have a ratio which is constant all over the two charts, and the frequency-function for chart C' is

$$f' = Af_0 e^{-\frac{1}{2}Q} \quad \text{.....(79),}$$

where

$$Q \equiv x_1'^2 + x_2'^2 + \ldots + x_n'^2 \quad \text{.....(80).}$$

The constant A is easily expressed in terms of the correlation coefficients r_{12}, r_{13}, ... by squaring the determinant of the transformation as in Art. 18 and using the relations (75) when we obtain

$$A^2 = \begin{vmatrix} 1, & r_{12}, & r_{13}, \ldots, & r_{1n} \\ r_{12}, & 1, & r_{23}, \ldots, & r_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ r_{1n}, & r_{2n}, & r_{3n}, \ldots, & 1 \end{vmatrix} \equiv R, \text{ say,} \quad \text{.....(81),}$$

therefore

$$A = R^{\frac{1}{2}} \quad \text{.....(82).}$$

We can now express the constant f_0 by setting

$$dF' = f'\,dx_1'\,dx_2' \ldots dx_n' \quad \text{.....(83),}$$

and integrating both sides over the entire chart C', that is between the limits $-\infty$ and $+\infty$ for each variable using (79), and taking the product of n separate integrals of the well-known type

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi} \quad \text{.....(84),}$$

and thus obtain for the total frequency

$$N = Af_0 (2\pi)^{\frac{n}{2}},$$

therefore

$$f_0 = \frac{N}{(2\pi)^{\frac{n}{2}} R^{\frac{1}{2}}} \quad \text{.....(85),}$$

30. *Fundamental lines of transformed chart.* Those hyperplanes of C' which correspond to the co-ordinate hyperplanes $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$ have the equations

$$\left.\begin{aligned} a_{11}x_1' + a_{12}x_2' + \ldots + a_{1n}x_n' &= 0 \\ a_{21}x_1' + a_{22}x_2' + \ldots + a_{2n}x_n' &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1' + a_{n2}x_2' + \ldots + a_{nn}x_n' &= 0 \end{aligned}\right\} \quad \text{.....(86),}$$

and will be denoted by P_1', P_2', ..., P_n'; and their n intersections which correspond to the axes OX_1, OX_2, ..., OX_n form an angloid which we denote, after the manner of Art. 11, by

$$\alpha' \equiv (L_1', L_2', \ldots, L_n') \quad \text{.....(87);}$$

the normals to the n hyperplanes, drawn as explained in that article, form the conjugate angloid which we denote by

$$\alpha \equiv (L_1, L_2, \ldots, L_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (88);$$

and the direction cosines of its n rays are, by Art. 16, proportional to

$$\left.\begin{array}{llll} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (89);$$

this will be called the *characteristic angloid* of the correlation in question.

31. *Relation between the face angles of the characteristic angloid and the coefficients of correlation.* We have from (75)

$$r_{pq} = a_{p1}a_{q1} + a_{p2}a_{q2} + \ldots + a_{pn}a_{qn} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (90),$$

hence, from (19)

$$r_{pq} = \cos(L_p, L_q) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (91),$$

thus the coefficients of correlation are represented by the cosines of the face angles of the characteristic angloid; hence the magnitudes of the face angles and of the dihedrals are independent of the particular transformation by which Q is resolved into a sum of squares, and depend only on the invariant quantities $r_{12}, r_{13}, \ldots$; a different mode of making this resolution would merely revolve the angloid about an axis passing through its vertex O. The most convenient method is considered in the next article.

32. *Canonical transformation of Q.* A particular mode of resolution which is of special interest consists in "completing the square" with regard to the different variables in succession: e.g.

$$Q = \frac{1}{c_{11}}\left[(c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n)^2 + (c_{11}c_{22} - c_{12}^2)x_2^2 + (c_{11}c_{33} - c_{13}^2)x_3^2 + \ldots \right.$$
$$\left. + 2(c_{11}c_{23} - c_{12}c_{13})x_2x_3 + \ldots\right] \quad \ldots\ldots (92);$$

here we have completed the square with regard to x_1, leaving a quadratic in the remaining variables; and we can similarly complete the square with regard to another variable, and so on; the resulting transformation is then

$$Q = x_1'^2 + x_2'^2 + \ldots + x_n'^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (93),$$

where the new variables have the form

$$\left.\begin{array}{rl} x_1' = & b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + \ldots + b_{1n}x_n \\ x_2' = & \quad b_{22}x_2 + b_{23}x_3 + \ldots + b_{2n}x_n \\ x_3' = & \quad b_{33}x_3 + \ldots + b_{3n}x_n \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' = & \quad b_{nn}x_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (94).$$

If we wish to express the old variables in terms of the new ones, as in (70) we can solve for x_n, x_{n-1}, ... in succession and obtain expressions of the form

$$\left.\begin{aligned} x_1 &= a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \ldots + a_{1n}x_n' \\ x_2 &= \qquad a_{22}x_2' + a_{23}x_3' + \ldots + a_{2n}x_n' \\ x_3 &= \qquad\qquad a_{33}x_3' + \ldots + a_{3n}x_n' \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= \qquad\qquad\qquad a_{nn}x_n' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(95).$$

It will be seen that, while we have taken the variables in the order $x_1, x_2, \ldots, x_n$, they could be taken in any other order, so that there are many ways of making this canonical resolution. The canonical transformation from the chart C to the chart C', based on (94), has the useful property that, while the axes OX_1, OX_2, ... do not in general correspond to the axes $O'X_1'$, $O'X_2'$, ..., yet any space S_m, determined by the first m of the axes OX_1, OX_2, ..., OX_m, corresponds to the space S_m' determined by the axes $O'X_1'$, $O'X_2'$, ..., $O'X_m'$, where m is any integer from 1 to n; for the $n-m$ equations of the planoid S_m are $x_n = 0$, $x_{n-1} = 0, \ldots, x_{m+1} = 0$, and from these follow, by (95), $x_n' = 0$, $x'_{n-1} = 0$, ..., $x'_{m+1} = 0$, which are the $n-m$ equations of the planoid S_m'. In particular the plane of the axes OX_1, and OX_2, corresponds to the plane of the axes $O'X_1'$ and $O'X_2'$. This correspondence will be used in the next article.

33. *Coefficients of partial correlation.* It is often useful to know the degree of correlation which exists between the deviations of the pth and qth characteristics, when the other characteristics are kept constant, that is when their deviations are kept at zero. The coefficient of correlation between x_p and x_q under these conditions is called their coefficient of *partial correlation* and is denoted by $r_{pq.12\ldots n}$, or by r_{pq} when there is no ambiguity as to what the other variables are. We shall select the two variables in question for our x_1 and x_2 and apply the canonical transformation of the last article. Since we have $x_3 = 0$, $x_4 = 0$, ..., and $x_n = 0$, hence we have also $x_3' = 0$, $x_4' = 0$, ..., and $x_n' = 0$, thus the only portions of the two charts with which we are now concerned are the plane of the axes OX_1, OX_2, and the plane of the axes $O'X_1'$, $O'X_2'$. The lines OX_1 and OX_2 correspond to lines $O'L_1$ and $O'L_2$ (lying in the latter plane) whose equations are, respectively,

$$\left.\begin{aligned} x_2 &\equiv a_{21}x_1' + a_{22}x_2' = 0 \\ x_1 &\equiv a_{11}x_1' + a_{12}x_2' = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(96),$$

thus $r_{12.34\ldots n}$, being equal to r_{12} on the two-dimensional chart, equals* $a_{11}a_{21} + a_{12}a_{22}$ by (75), which is evidently the cosine of the angle between the normals to L_1 and L_2, and this angle is supplemental to the angle between L_1 and L_2, and is hence equal to the dihedral (P_1, P_2) as in (4), therefore

$$r_{12.34\ldots n} = \cos \operatorname{supp}(L_1, L_2) = \cos(P_1, P_2),$$

* For symmetry we have retained a_{21}, although it happens to be zero in this transformation.

and, as any pair of the variables could have been selected for x_1 and x_2, we have in general

$$r_{pq.12\ldots n} = \cos(P_p, P_q) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(97).$$

The relations between the coefficients of total* and of partial correlation may accordingly be stated geometrically as follows:

If the coefficients of total correlation be represented by the cosines of the face angles of an n-dimensioned angle, then the coefficients of partial correlation will be represented by the cosines of the respective dihedral angles between the hyperplanes obtained by omitting each ray in turn†.

This geometric relation will be used in connection with the goniometry of the n-dimensioned angle to obtain an algebraic expression for any coefficient of partial correlation in terms of all the coefficients of total correlation $r_{12}, r_{13}, \ldots$.

34. *Partial correlation coefficients in terms of total.* Using (22), (25), (81), (97), we have at once

$$r_{pq.12\ldots n} = \frac{-R_{pq}}{(R_{pp}R_{qq})^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(98),$$

where the determinants on the right are minors of the determinant

$$R \equiv \begin{vmatrix} 1, & r_{12}, & r_{13}, & \ldots, & r_{1n} \\ r_{12}, & 1, & r_{23}, & \ldots, & r_{2n} \\ r_{13}, & r_{23}, & 1, & \ldots, & r_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ r_{1n}, & r_{2n}, & r_{3n}, & \ldots, & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(99),$$

in which the coefficients of "total correlation" are to be obtained by the ordinary formula for the correlation of two characteristics

$$r_{pq} = \frac{\Sigma x_p x_q}{N},$$

in which we consider only the pair of deviations x_p and x_q for each of the N individuals, and entirely ignore any other characteristics‡.

35. *Total correlation coefficients in terms of partial.* From (27), (28), (97) we have

$$r_{pq} = \frac{\mathfrak{R}_{pq}}{(\mathfrak{R}_{pp}\mathfrak{R}_{qq})^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(100),$$

* The ordinary coefficient of correlation r_{pq}, computed from the N pairs of values of x_p and x_q without observing or restricting any other variables, is called the coefficient of total correlation between the pth and qth characteristics, to distinguish it from a coefficient which is obtained by imposing restrictions on any of the variables.

† A corresponding representation for the coefficients of multiple correlation will be given in Art. 42.

‡ It will be noticed that, in the case of partial correlation, the remaining characteristics are not ignored, but are maintained constant. We shall see later that there are certain intermediate coefficients in which some of the characteristics are ignored and others kept constant.

in which the determinants are minors of the determinant

$$\mathfrak{R} \equiv \begin{vmatrix} 1, & -r_{12}, & -r_{13}, & \ldots, & -r_{1n} \\ -r_{12}, & 1, & -r_{23}, & \ldots, & -r_{2n} \\ -r_{13}, & -r_{23}, & 1, & \ldots, & -r_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ -r_{1n}, & -r_{2n}, & -r_{3n}, & \ldots, & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots(101),$$

where r_{pq} is an abbreviation for $r_{pq.12\ldots n}$.

36. *The coefficients of Q in terms of r_{12}, r_{13}, ...* Solving (70) we obtain

$$\left.\begin{aligned} Ax_1' &= A_{11}x_1 + A_{21}x_2 + \ldots + A_{n1}x_n \\ Ax_2' &= A_{12}x_1 + A_{22}x_2 + \ldots + A_{n2}x_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Ax_n' &= A_{1n}x_1 + A_{2n}x_2 + \ldots + A_{nn}x_n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(102),$$

where A is the determinant of the a's; then substituting in (69), we have

$$\begin{aligned} A^2Q = (A_{11}^2 + A_{12}^2 + \ldots)x_1^2 + (A_{21}^2 + A_{22}^2 + \ldots)x_2^2 + \ldots \\ + 2(A_{11}A_{21} + A_{12}A_{22} + \ldots)x_1x_2 + \ldots \ldots\ldots\ldots(103), \end{aligned}$$

but these coefficients are the invariants mentioned in Art. 18, hence by Art. 19

$$Q = \frac{1}{R}(R_{11}x_1^2 + R_{22}x_2^2 + \ldots + 2R_{12}x_1x_2 + \ldots) \quad \ldots\ldots\ldots\ldots(104).$$

37. *Regression locus of x_p on the other characteristics.* If a line l_p be drawn on the chart C parallel to the axis OX_p, the successive points through which it passes have of course different values of the frequency function attached to them; this distribution of frequency follows the Gaussian law with regard to the single variable x_p, as may be readily seen by keeping the other variables constant in (63), and letting x_p alone vary, in which case f can be put in the form

$$f_p = f_0 e^{-(ax_p^2 + 2bx_p + c)} = f_0 e^{-a\left[\left(x_p + \frac{b}{a}\right)^2 + \frac{ac - b^2}{a^2}\right]} \quad \ldots\ldots\ldots\ldots(105),$$

which is a Gaussian function whose maximum value occurs at $x_p = -\dfrac{b}{a}$, which is called its *mode*, and whose modal value is

$$f_{p0} = f_0 e^{\frac{-ac + b^2}{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(106),$$

where a, b, c depend on the assigned constant values of the other $n-1$ variables. When the line l_p is taken in successive positions, remaining parallel to OX_p, we shall show that the mode of the Gaussian function describes a hyperplane, for it is clear that f_{p0} has its maximum value where Q has its minimum, that is where $\dfrac{\partial Q}{\partial x_p} = 0$, hence the locus of the mode is

$$R_{p1}x_1 + R_{p2}x_2 + \ldots + R_{pn}x_n = 0 \ldots\ldots\ldots\ldots\ldots (107);$$

this hyperplane is called the *regression locus* of x_p on the other characteristics. It may be defined, in accordance with the statements above, as the locus of points of

maximum frequency on a series of lines drawn parallel to the axis OX_p. The value of x_p given by this equation in terms of the remaining variables is called its *regression value* and is denoted by $\bar{x}_p$, thus

$$\bar{x}_p = -\frac{R_{p1}x_1 + R_{p2}x_2 + \ldots + R_{pn}x_n}{R_{pp}} \text{ omitting } x_p \quad \ldots\ldots\ldots\ldots(108);$$

this is the most probable value of x_p when the other deviations have assigned values, because it is that value of x_p which most frequently accompanies the assigned values of the other deviations; it is also the average of all the values of x_p which can accompany those assigned values. This regression value is the best prediction that can be made as to the magnitude of an unknown x_p when its correlations with regard to known characteristics are given. A useful expression for the mean departure from such prediction will now be given.

38. *Scatter of the values of x_p from its regression locus.* The *scatter* S_p of the variable x_p is defined as the root-mean-square of the departures of the actual values of x_p from its regression value $\bar{x}_p$ corresponding to assigned values of the other $n-1$ variables.

We shall first express S_1; and we begin by writing Q as a quadratic function of x_1 and completing the square in the form

$$Q = \frac{R_{11}}{R}\left[\left(x_1 + \frac{R_{12}x_2 + R_{13}x_3 + \ldots}{R_{11}}\right)^2 + \ldots\right] \quad \ldots\ldots\ldots\ldots(109);$$

hence by (108), the frequency function is of the form

$$f = Ce^{\frac{-R_{11}}{2R}(x_1 - \bar{x}_1)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(110),$$

which is of the familiar normal type

$$f = Ce^{\frac{-(x_1 - \bar{x}_1)^2}{2S_1^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(111),$$

in which S_1 is the root-mean-square of the various values of the departure $x_1 - \bar{x}_1$; then equating corresponding constant coefficients we have

$$\frac{R_{11}}{2R} = \frac{1}{2S_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(112),$$

therefore

$$S_1 = \left(\frac{R}{R_{11}}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(113).$$

In a similar way

$$S_p = \left(\frac{R}{R_{pp}}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(114),$$

which may also be written

$$\frac{1}{N}\Sigma(x_p - \bar{x}_p)^2 = \frac{R}{R_{pp}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(115).$$

It is interesting to see that this expression for S_p is independent of the assigned values of the remaining variables, and is thus the same for all the columns of the chart which are formed by lines parallel to the axis OX_p. It follows that on a normal correlation chart, the values of x_p are equally scattered about its regression

locus in all columns parallel to OX_p; and the distribution is then said to be *homoscedastic*.

39. *Probable error of predictions based on correlation.* We have seen that the best prediction as to the magnitude of x_p when the other characteristics are assigned is the value $\bar{x}_p$ furnished by the regression equation. We are now able, by means of the scatter, to give an expression for the probable error of such prediction, and thus complete in the usual form the best statistical determination obtainable from the data under the hypothesis of normal correlation.

From the theory of errors, the probable error is ·6745 times the mean error*, thus

$$E_p = \cdot 6745\, S_p \quad \ldots\ldots(116),$$

hence the best statistical determination for x_p, when the other deviations and the correlations are given, is

$$x_p = \bar{x}_p \pm \cdot 6745\, S_p \quad \ldots\ldots(117),$$

where $\bar{x}_p$ and S_p are given by (108) and (114) respectively; the first term is the most probable value of x_p and is the best prediction; the second term is that departure which is as likely to be exceeded as not, and the odds are against the departure exceeding any assigned value larger than this. Stated in terms of frequency of successes, we can say that 50 per cent. of the predictions based on (117) would have an error less than ·6745 S_p, that 82 per cent. would have an error less than twice that term, and 97 per cent. an error less than three times that amount; thus the error of the prediction practically never exceeds $2S_p$, and is usually less than S_p.

40. *Graphical representation of* S_p. We have by (48)

$$\sin \pi_p = \left(\frac{R}{R_{pp}}\right)^{\frac{1}{2}},$$

where π_p is the perpendicular arc drawn from the vertex p on the hypersphere to the opposite side of the hyperspherical polygon determined by the characteristic angloid. Hence

$$S_p = \sin \pi_p \quad \ldots\ldots(118).$$

41. *Comparison of probable errors in different methods of prediction.* The value of x_p can of course be predicted from the values of any number m of the other variables by ignoring all but these $m+1$ variables, and confining the chart to $m+1$ dimensions. We shall show, however, that this will increase the probable error of the prediction. To prove this, let us ignore the qth characteristic, and denote the scatter of x_p from the corresponding regression locus by $S_{p,(q)}$, then

$$S_{p,(q)} = \sin \pi_{p,(q)} \quad \ldots\ldots(119),$$

where $\pi_{p,(q)}$ is the perpendicular arc drawn from the point p on the hypersphere to the hyper-great-circle on which lie the remaining vertices. Now from (50)

$$\frac{\sin \pi_p}{\sin \pi_{p,(q)}} = \sin (p', q') = \sin (P_p, P_q) \quad \ldots\ldots(120),$$

* This is Gauss' term for the root-mean-square of all possible errors.

therefore

$$\frac{S_p}{S_{p,(q)}} = (1 - r^2_{pq.12\ldots n})^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(121),$$

thus $S_{p,(q)} > S_p$, unless r_{pq} is zero, that is, unless there is no correlation between p and q, when the other variables are kept constant. Thus by (121) the probable error of the prediction is in general increased by ignoring any characteristic; and it is still further increased by ignoring more of the characteristics. Hence the best prediction is the one that makes use of all the known correlations.

It may happen, however, that a small number of the characteristics will furnish nearly as good a prediction as a larger number would furnish, especially if the ignored characteristics are not very closely correlated with the characteristic in question.

The following actual example will illustrate this point, and will serve to show how the regression formula and its probable error are computed numerically.

Two hundred college students whose high school averages were known, and whose first year college averages were also known, were subjected to the standard "intelligence test." We shall denote these three tests by G, H, I, and number them as 1, 2, 3, making the numerical order correspond to the alphabetical. The average of the college grades was $\bar{x}_1 = 75$, with a standard deviation of $\sigma_1 = 9{\cdot}5$; the average of the high school grades was $\bar{x}_2 = 80$, with $\sigma_2 = 5{\cdot}8$; and the average of the intelligence grades was $\bar{x}_3 = 59$, with $\sigma_3 = 12{\cdot}8$; the three coefficients of total correlation, computed from the individual deviations by the usual product-moment formula, were $r_{12} = {\cdot}47$, $r_{13} = {\cdot}31$, $r_{23} = {\cdot}37$; we wish to find the best statistical formula for the deviation x_1 when the deviations x_2 and x_3 are assigned.

The regression value of x_1 is to be obtained from the equation

$$R_{11}x_1 + R_{12}x_2 + R_{13}x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(122),$$

these coefficients being minors of the determinant R,

$$R = \begin{vmatrix} 1, & r_{12}, & r_{13} \\ r_{21}, & 1, & r_{23} \\ r_{31}, & r_{32}, & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(123),$$

R_{ij} being minor of element r_{ij} ($r_{ii} = 1$).

If, however, we ignore the third characteristic the regression value of x_1 is to be found from the equation

$$x_1 - r_{12}x_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(124).$$

Again, if we ignore the second characteristic we have, for the regression equation,

$$x_1 - r_{13}x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(125).$$

Comparison of the three probable errors shows that nearly as good a prediction can be made for x_1 from x_2 alone as from x_2 and x_3 together; the reason

for this is that the partial correlation between x_1 and x_3 when x_2 is kept constant is comparatively small, viz.

$$r_{13.2} = \cdot 17 \quad \ldots\ldots\ldots\

directly with the coefficients of correlation. The degree of closeness between x_p and $\bar{x}_p$ is best indicated by their own coefficient of correlation:

$$r(x_p, \bar{x}_p) \equiv \frac{\Sigma x_p \bar{x}_p}{\sqrt{(\Sigma x_p^2)(\Sigma \bar{x}_p^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (130);$$

to evaluate this we employ the two useful relations

$$\Sigma \bar{x}_p (x_p - \bar{x}_p) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (131),$$

$$\Sigma x_p (x_p - \bar{x}_p) = \frac{NR}{R_{pp}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (132),$$

the first of which is evident from the fact that, for an assigned value of $\bar{x}_p$, the departure $x_p - \bar{x}_p$ has a normal distribution about the value zero, and is thus as often positive as negative; and the second follows by putting (115) in the form

$$\Sigma x_p (x_p - \bar{x}_p) - \Sigma \bar{x}_p (x_p - \bar{x}_p) = \frac{NR}{R_{pp}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (133),$$

then from

$$\Sigma \bar{x}_p^2 = \Sigma x_p \bar{x}_p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (134),$$

$$\Sigma x_p \bar{x}_p = \Sigma x_p^2 - \frac{NR}{R_{pp}} = N\left(1 - \frac{R}{R_{pp}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (135),$$

hence (130) reduces to

$$r(x_p, \bar{x}_p) = \left(1 - \frac{R}{R_{pp}}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (136).$$

Since this coefficient indicates how closely x_p can be expressed as a linear function of the other variables, it is called the coefficient of correlation between x_p and the group of the remaining characteristics, and is denoted by $r_{p,12\ldots n}$, or by r_p when there is no ambiguity as to the remaining characteristics. Indices of this class are known as *coefficients of multiple correlation.*

43. *Relation between r_p and S_p.* From (48), (118) and (136) we have

$$r_{p,12\ldots n} = (1 - S_p^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (137).$$

As a verification it may be noticed that this relation accords with the fact that when $S_p = 0$, every value of x_p is given accurately by the regression equation in terms of the other variables, and there is a perfect correlation between x_p and $\bar{x}_p$; and it accords with the further fact that when r_p is zero, the regression locus is horizontal (that is, perpendicular to the axis OX_p), in which case the scatter of x_p is identical with σ_p, and the scatter of x_p is unity.

As S_p is represented by the sine of the inclination of the ray L_p of the characteristic angloid to the hyperplane of the other rays, it follows that r_p is represented by the cosine of the same inclination, or, on the surface of the hypersphere, by the cosine of the perpendicular arc from the vertex p to the hyper-great-circle P_p; that is

$$r_{p,12\ldots n} = \cos \pi_p \quad \ldots\ldots\ldots\ldots\ldots\ldots (138).$$

44. *Comparison of multiple coefficients of different orders.* The value of the coefficient of correlation of x_p on the group of the remaining variables is reduced when we ignore any of the latter; for we have shown in Art. 41 that the scatter

is increased; and then, from (137), the coefficient of multiple correlation is accordingly diminished. By continuing the process of ignoring characteristics we arrive at a coefficient of total correlation; hence $r_{p,12\dots n}$ is greater than any of the total coefficients $r_{p1}, r_{p2}, \dots$, or at least equal to the greatest of them. It is likewise not less than the greatest of the partial coefficients $r_{p1}, r_{p2}, \dots$, for we see from (121) and (137) that

$$\frac{1-r^2_{p,12\dots n}}{1-r^2_{p,12\dots(q)\dots n}} = 1-r^2_{pq.12\dots n} \quad \dots\dots\dots(139),$$

therefore

$$r_{p,12\dots n} \nless r_{pq.12\dots n} \quad \dots\dots\dots(140).$$

To illustrate we may compare $r_{1,23}$ with r_{12} and r_{13} in Art. 41, we have

$$r_{1,23} = (1-S^2_{1,23})^{\frac{1}{2}} = (1-(\cdot 87)^2)^{\frac{1}{2}} = \cdot 49,$$

while

$$r_{12} = \cdot 47, \quad r_{13} = \cdot 31;$$

thus we see again that nearly as good a prediction for x_1 can be obtained from x_2 as from the group consisting of x_2 and x_3. The corresponding partial correlations are also smaller than $r_{1,23}$, for we saw that

$$r_{12.3} = \cdot 40, \quad r_{13.2} = \cdot 17.$$

If we ignore another variable x_s we have

$$\frac{1-r^2_{p,12\dots(q)\dots n}}{1-r^2_{p,12\dots(qs)\dots n}} = 1-r^2_{ps.12\dots(q)\dots n} \quad \dots\dots\dots(141),$$

hence, by multiplication,

$$\frac{1-r^2_{p,12\dots n}}{1-r^2_{p,12\dots(qs)\dots n}} = (1-r^2_{pq.12\dots n})(1-r^2_{ps.12\dots(q)\dots n}) \quad \dots\dots\dots(142).$$

If we continue this process we obtain a series of useful relations like (142) the final one of which can be put in the convenient form

$$\frac{1-r^2_{p,12\dots n}}{1-r_{p1}{}^2} = (1-r^2_{p2.1})(1-r^2_{p3.12})(1-r^2_{p4.123})\dots(1-r^2_{pn.12\dots\overline{n-1}}) \quad \dots(143),$$

where we have ignored in succession the variables $x_n, x_{n-1}, \dots$, and we have reversed the order of the factors on the right.

45. *Relations among partial coefficients of different orders.* A partial correlation coefficient obtained by taking all of the n characteristics into account (two of them being allowed to vary and the other $n-2$ being kept constant) is said to be of the nth *order*; it is, as we have seen in Arts. 30, 31, represented by the cosine of a dihedral of the characteristic n-dimensioned angle

$$\alpha = (L_1, L_2, \dots, L_n) \quad \dots\dots\dots(144),$$

that is, in the notation explained in Arts. 11, 12, we have

$$r_{pq.12\dots n} = \cos[L_1 L_2 \dots (L_p) \dots L_n,\ L_1 L_2 \dots (L_q) \dots L_n] = \cos(P_p, P_q) \quad \dots(145).$$

If, however, we ignore one of the characteristics X_s, we must omit the corresponding ray L_s, and confine our attention to the $(n-1)$-dimensioned angle

$$\alpha_s = [L_1, L_2, \dots (L_s) \dots, L_n] \quad \dots\dots\dots(146),$$

and the coefficient of partial correlation between x_p and x_q, when x_s is ignored and the remaining characteristics are kept constant, is represented by the cosine of the dihedral (P_{ps}, P_{qs}), in the notation of Art. 11. Thus

$$r_{pq.12\ldots(s)\ldots n} = \cos(P_{ps}, P_{qs}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (147);$$

this is called a partial coefficient of order $n-1$. Similarly if we ignore two characteristics x_s and x_t we have

$$r_{pq.12\ldots(st)\ldots n} = \cos(P_{pst}, P_{qst}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (148),$$

a coefficient of order $n-2$.

Thus to obtain relations among coefficients of different orders we have only to use the relations, obtained in Art. 22, among dihedrals of different orders.

Taking (30) and replacing the cosines of dihedrals by the corresponding coefficients of correlation, we have

$$r_{pq.12\ldots(s)\ldots n} = \frac{r_{pq} + r_{ps}r_{qs}}{(1-r_{ps}^2)^{\frac{1}{2}}(1-r_{qs}^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (149).$$

This expresses a coefficient of order $n-1$ in terms of coefficients of order n. If we ignore another variable on both sides of (149) we get a coefficient of order $n-2$ on the left, and three coefficients of order $n-1$ on the right; thus $r_{pq.(st)\ldots n}$ can be expressed as in (35) in terms of the three coefficients $r_{pq.(t)\ldots n}$, $r_{ps.(t)\ldots n}$, $r_{qs.(t)\ldots n}$; we can similarly express a coefficient of *any* order in terms of coefficients whose order is one higher.

Next to express a coefficient of any order in terms of coefficients whose order is two units higher, we take equation (148) in which two variables are absent on the left, then by (31)

$$r_{pq.(st)\ldots n} = \frac{\mathfrak{R}^{\text{iv}}_{pq}}{\sqrt{\mathfrak{R}^{\text{iv}}_{pp}\,\mathfrak{R}^{\text{iv}}_{qq}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (150),$$

in which the determinants on the right are minors of the fourth order determinant

$$\mathfrak{R}^{\text{iv}} \equiv \begin{vmatrix} 1 & , -r_{pq.} & , -r_{ps.} & , -r_{pt.} \\ -r_{pq.} & 1 & , -r_{qs.} & , -r_{qt.} \\ -r_{ps.} & , -r_{qs.} & 1 & , -r_{st.} \\ -r_{pt.} & , -r_{qt.} & , -r_{st.} & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (151).$$

which is formed from the pth, qth, sth, and tth rows of the determinant $\mathfrak{R}$ in Art. 35, and from the pth, qth, sth, and tth columns of the same determinant. If we ignore another variable on both sides of (148) we get a coefficient of order $n-3$ on the left and six coefficients of order $n-1$ on the right, thus $r_{pq.(stu)\ldots n}$ can be expressed in terms of the six coefficients

$$r_{pq.(u)\ldots n},\ r_{ps.(u)\ldots n},\ r_{pt.(u)\ldots n}, \cdots$$

in the same form as (150).

To express a coefficient of any order in terms of coefficients whose order is three units higher we take (148) in which three variables are absent on the left, then by (33)

$$r_{pq\,.(stu)\ldots n} = \frac{\mathfrak{R}^{v}_{pq}}{(\mathfrak{R}^{v}_{pp}\,\mathfrak{R}^{v}_{qq})^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(152),$$

in which the determinants are minors of the fifth order determinant $\mathfrak{R}^{v}$ obtained by taking the pth, qth, sth, tth, and uth rows and columns of the determinant $\mathfrak{R}$ in Art. 35. In this, as before, other variables can be ignored all through, on both sides of the equation. The rule indicated in (33) can evidently be applied to coefficients of any order.

It is sometimes desired to reverse this process, and to express a coefficient of any order in terms of coefficients whose order is one or more units lower.

From the relation (34) among dihedrals we have at once

$$r_{pq} = \frac{r_{pq\,.\,(s)}\cdots r_{ps\,.\,(q)}\, r_{qs\,.\,(p)}}{(1-r^{2}_{ps\,(q)})^{\frac{1}{2}}(1-r^{2}_{qs\,.\,(p)})^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(153),$$

in which $r_{pq\,.\,(s)}$ is an abbreviation for $r_{pq\,.(s)\ldots n}$, and similarly for the other symbols on the right; these are coefficients of order $n-1$, while r_{pq} is of order n. If we ignore other subscripts on both sides we have an expression for a coefficient of any order in terms of coefficients whose order is one unit lower. We next take (35) and convert it into a relation among coefficients of correlation, and obtain

$$r_{pq} = \frac{S_{pq}}{(S_{pp}S_{qq})^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(154),$$

in which the determinants are minors of the determinant*

$$\begin{vmatrix} 1 & r_{pq.\,(st)}, & r_{ps.\,(qt)}, & r_{pt.\,(qs)} \\ r_{pq.\,(st)}, & 1 & r_{qs.\,(pt)}, & r_{qt.\,(ps)} \\ r_{ps.\,(qt)}, & r_{qs.\,(pt)}, & 1 & r_{st.\,(pq)} \\ r_{pt.\,(qs)}, & r_{qt.\,(ps)}, & r_{st.\,(pq)}, & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(155),$$

the minor S_{pq} being obtained by erasing the row in which p is among the first two letters, and the column in which q is among the first two letters. By ignoring any other assigned subscripts on both sides we can thus express a coefficient of *any* order in terms of coefficients whose order is two units lower.

The generality of this process is obvious; and if we proceed to express r_{pq} in terms of coefficients whose order is m units lower the order of the determinant S will evidently be $m+2$, and its minors will be of order $m+1$, and the same rule of formation applies.

* Here $r_{pq.\,(st)}$ is an abbreviation for $r_{pq.\,(st)\ldots n}$.

ON THE GENERAL FORMS OF BIVARIATE FREQUENCY DISTRIBUTIONS WHICH ARE MATHEMATICALLY POSSIBLE WHEN REGRESSION AND VARIATION ARE SUBJECTED TO LIMITING CONDITIONS.

By Professor SEIMATSU NARUMI.

PART II (*to be continued**).

(4) *Complete Homoscedasticity.*

We have already found (Part I, p. 83) Equations (ix) and (ix) *bis*, namely:

$$\frac{\psi_1'''\{x-f_1(y)\}}{\psi_1''\{x-f_1(y)\}}=\frac{f_2''(x)}{f_2'(x)}-f_2'(x)\frac{\psi_2'''\{y-f_2(x)\}}{\psi_2''\{y-f_2(x)\}} \quad \text{.....(ix),}$$

and

$$\frac{f_1''(y)}{f_1'(y)}-f_1'(y)\frac{\psi_1'''\{x-f_1(y)\}}{\psi_1''\{x-f_1(y)\}}=\frac{\psi_2'''\{y-f_2(x)\}}{\psi_2''\{y-f_2(x)\}} \quad \text{..... (ix) } bis.$$

Now differentiate Equation (ix) with regard to y, or Equation (ix) *bis* with regard to x; we have

$$f_1'(y)\frac{\psi_1^{iv}(u)\,\psi_1''(u)-\psi_1'''(u)^2}{\psi_1''(u)^3}=f_2'(x)\frac{\psi_2^{iv}(v)\,\psi_2''(v)-\psi_2'''(v)^2}{\psi_2''(v)^3},$$

where $$u=x-f_1(y) \quad \text{and} \quad v=y-f_2(x).$$

By virtue of Equation (iii), this may be rewritten in the form

$$\frac{\psi_1^{iv}(u)\,\psi_1''(u)-\psi_1'''(u)^2}{\psi_1''(u)^3}=\frac{\psi_2^{iv}(v)\,\psi_2''(v)-\psi_2'''(v)^2}{\psi_2''(v)^3},$$

which must be a constant, say 2ϵ, excepting the case in which

$$f_1'(y)f_2'(x)=1.$$

Thus we have

$$\psi_1^{iv}(u)\,\psi_1''(u)-\psi_1'''(u)^2=2\epsilon\psi_1''(u)^2,$$

and

$$\psi_2^{iv}(v)\,\psi_2''(v)-\psi_2'''(v)^2=2\epsilon\psi_2''(v)^2.$$

By integration we find

$$\left\{\frac{\psi_1'''(u)}{\psi_1''(u)}\right\}^2=2\epsilon\psi_1''(u)+a \quad \text{.....(x),}$$

$$\left\{\frac{\psi_2'''(v)}{\psi_2''(v)}\right\}^2=2\epsilon\psi_2''(v)+c \quad \text{.....(x) } bis,$$

where a and c are constants.

* See *Biometrika*, Vol. xv. pp. 77—88 for Part I.

Now transferring the terms in Equations (ix) and (ix) *bis* we have

$$\frac{\psi_1'''(u)}{\psi_1''(u)} + f_2'(x)\frac{\psi_2'''(v)}{\psi_2''(v)} = \frac{f_2''(x)}{f_2'(x)},$$

$$\frac{\psi_2'''(v)}{\psi_2''(v)} + f_1'(y)\frac{\psi_1'''(u)}{\psi_1''(u)} = \frac{f_1''(y)}{f_1'(y)},$$

and squaring and dividing by $f_2'(x)$ and $f_1'(y)$ respectively, we find

$$2e\left(\frac{\psi_1''(u)}{f_2'(x)} + f_2'(x)\psi_2''(v)\right) + 2\frac{\psi_1'''(u)}{\psi_1''(u)}\cdot\frac{\psi_2'''(v)}{\psi_2''(v)} + \frac{a}{f_2'(x)} + cf_2'(x) = \frac{f_2''(x)^2}{f_2'(x)^2},$$

$$2e\left(f_1'(y)\psi_1''(u) + \frac{\psi_2''(v)}{f_1'(y)}\right) + 2\frac{\psi_1'''(u)}{\psi_1''(u)}\cdot\frac{\psi_2'''(v)}{\psi_2''(v)} + af_1'(y) + \frac{c}{f_1'(y)} = \frac{f_1''(y)^2}{f_1'(y)^2},$$

or transferring terms:

$$2e\left(\frac{\psi_1''(u)}{f_2'(x)} + f_2'(x)\psi_2''(v)\right) + 2\frac{\psi_1'''(u)}{\psi_1''(u)}\cdot\frac{\psi_2'''(v)}{\psi_2''(v)} = \frac{f_2''(x)^2}{f_2'(x)^2} - \frac{a}{f_2'(x)} - cf_2'(x),$$

$$2e\left(f_1'(y)\psi_1''(u) + \frac{\psi_2''(v)}{f_1'(y)}\right) + 2\frac{\psi_1'''(u)}{\psi_1''(u)}\cdot\frac{\psi_2'''(v)}{\psi_2''(v)} = \frac{f_1''(y)^2}{f_1'(y)^2} - af_1'(y) - \frac{c}{f_1'(y)}.$$

Remembering that $f_1'(y)\psi_1''(u) = f_2'(x)\psi_2''(v)$,

we see that the left-hand sides of these equations are equal, and therefore the right-hand sides must be equal to each other; but the former is a function of x only, while the latter is a function of y only. Accordingly they must each be equal to the same constant, say $2b$. Hence we have

$$f_1''(y)^2 = f_1'(y)^2(af_1'(y)^2 + 2bf_1'(y) + c) \quad \text{.....(xi)},$$

$$f_2''(x)^2 = f_2'(x)^2(a + 2bf_2'(x) + cf_2'(x)^2) \quad \text{.....(xi) } bis.$$

Case (i). $a > 0, \quad c > 0, \quad b^2 - ac \neq 0.$

Integrating Equation (xi) *bis* we shall have

$$f_2'(x) = \frac{am_2\left(\coth^2\frac{\sqrt{a}(x-\bar{x})}{2} - 1\right)}{\left(\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2} + n_2\right)^2 - m_2^2 c},$$

where m_2 is given by the initial condition that the regression curve at the mean of whole population will have

$$f_2'(\bar{x}) = m_2,$$

and n_2 is given by the relation

$$n_2^2 = a + 2bm_2 + cm_2^2.$$

Similarly from Equation (xi) we have

$$f_1'(y) = \frac{cm_1\left(\coth^2\frac{\sqrt{c}(y-\bar{y})}{2} - 1\right)}{\left(\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2} + n_1\right)^2 - m_1^2 a},$$

where

$$f_1'(\bar{y}) = m_1,$$

and n_1 is given by the relation

$$n_1^2 = am_1^2 + 2bm_1 + c.$$

By further integration we have

$$f_1(y) = \bar{x} + \frac{2}{\sqrt{a}} \coth^{-1} \left\{ \frac{\sqrt{c}}{m_1\sqrt{a}} \coth \frac{\sqrt{c}\,(y-\bar{y})}{2} + \frac{n_1}{m_1\sqrt{a}} \right\} \quad \text{.....(xii),}$$

and

$$f_2(x) = \bar{y} + \frac{2}{\sqrt{c}} \coth^{-1} \left\{ \frac{\sqrt{a}}{m_2\sqrt{c}} \coth \frac{\sqrt{a}\,(x-\bar{x})}{2} + \frac{n_2}{m_2\sqrt{c}} \right\} \quad \text{.....(xii) } \textit{bis},$$

from which we have

$$u = x - f_1(y) = x - \bar{x} - \frac{2}{\sqrt{a}} \coth^{-1} \left\{ \frac{\sqrt{c}}{m_1\sqrt{a}} \coth \frac{\sqrt{c}\,(y-\bar{y})}{2} + \frac{n_1}{m_1\sqrt{a}} \right\} \quad \text{.....(xiii),}$$

$$v = y - f_2(x) = y - \bar{y} - \frac{2}{\sqrt{c}} \coth^{-1} \left\{ \frac{\sqrt{a}}{m_2\sqrt{c}} \coth \frac{\sqrt{a}\,(x-\bar{x})}{2} + \frac{n_2}{m_2\sqrt{c}} \right\} \quad \text{...(xiii) } \textit{bis}.$$

Evidently, at the mean of whole population, we have

$$u = 0, \quad v = 0.$$

We are now in a position to find the functions $\psi_1(u)$ and $\psi_2(v)$. If we assume that

$$\frac{\psi_1'''(0)}{\psi_1''(0)} = \epsilon_1, \quad \frac{\psi_2'''(0)}{\psi_2''(0)} = \epsilon_2;$$

and

$$\psi_1''(0) = \delta_1, \quad \psi_2''(0) = \delta_2,$$

we must have, by Equation (iii),

$$m_1\delta_1 = m_2\delta_2 = m_1m_2\delta,$$

where δ is another constant. Therefore

$$\delta_1 = m_2\delta, \quad \delta_2 = m_1\delta;$$

and further

$$a = \epsilon_1^2 - 2m_2\epsilon\delta,$$

$$b = \epsilon_1\epsilon_2 + (1 + m_1m_2)\,\epsilon\delta,$$

$$c = \epsilon_2^2 - 2m_1\epsilon\delta,$$

$$n_1 = -(m_1\epsilon_1 + \epsilon_2),$$

$$n_2 = -(\epsilon_1 + m_2\epsilon_2).$$

Now, from Equation (x),

$$\frac{\psi_1'''(u)^2}{\psi_1''(u)^2} = 2\epsilon\psi_1''(u) + a,$$

by integration we have

$$\psi_1''(u) = \frac{2a}{\epsilon} \cdot \frac{e^{\sqrt{a}\,(u+c_1)}}{(1 - e^{\sqrt{a}\,(u+c_1)})^2}.$$

In order to determine a constant c_1, let us take the logarithmic differential of the last equation

$$\frac{\psi_1'''(u)}{\psi_1''(u)} = \sqrt{a} \left(\frac{1 + e^{\sqrt{a}\,(u+c_1)}}{1 - e^{\sqrt{a}\,(u+c_1)}} \right),$$

and put $u=0$. Then we have

$$\epsilon_1 = -\sqrt{a}\coth\frac{\sqrt{a}\,c_1}{2}.$$

Now, using Equation (xiii), we have

$$\psi_1''(u)=\frac{m_2 a\delta\left\{\coth^2\frac{\sqrt{a}(x-\bar{x})}{2}-1\right\}\left\{\left(\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2}+n_1\right)^2-m_1^2 a\right\}}{\left\{\sqrt{ac}\coth\frac{\sqrt{a}(x-\bar{x})}{2}\coth\frac{\sqrt{c}(y-\bar{y})}{2}-\epsilon_1\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2}-\epsilon_2\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2}+\epsilon_1\epsilon_2+2m_1m_2\epsilon\delta\right\}^2}.$$

Similarly from Equation (xiii) *bis*, by integrations, we have

$$\psi_2''(v)=\frac{m_1 c\delta\left\{\coth^2\frac{\sqrt{c}(y-\bar{y})}{2}-1\right\}\left\{\left(\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2}+n_2\right)^2-m_2^2 c\right\}}{\left\{\sqrt{ac}\coth\frac{\sqrt{a}(x-\bar{x})}{2}\coth\frac{\sqrt{c}(y-\bar{y})}{2}-\epsilon_1\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2}-\epsilon_2\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2}+\epsilon_1\epsilon_2+2m_1m_2\epsilon\delta\right\}^2}.$$

From these equations and Equations (xi) and (xi) *bis* we see that Equation (iii) will be satisfied for all values of x and y.

By integration we have

$$\begin{aligned}\psi_1(u) &= \gamma_1' + \gamma_1 u - \frac{2}{\epsilon}\left\{\log\left(\frac{1-e^{\sqrt{a}(u+c_1)}}{1-e^{\sqrt{a}c_1}}\right)+\frac{\sqrt{a}e^{\sqrt{a}c_1}}{1-e^{\sqrt{a}c_1}}u\right\}\\ &= \gamma_1' + \gamma_1 u - \frac{2}{\epsilon}\log\left\{\sqrt{ac}\coth\frac{\sqrt{a}(x-\bar{x})}{2}\coth\frac{\sqrt{c}(y-\bar{y})}{2}\right.\\ &\qquad\left.-\epsilon_1\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2}-\epsilon_2\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2}+\epsilon_1\epsilon_2+2m_1m_2\epsilon\delta\right\}\\ &\qquad-\frac{2}{\epsilon}\left\{\log\left(\frac{\sinh\frac{\sqrt{a}(x-\bar{x})}{2}}{\sqrt{a}}\right)-\log\left(\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2}+n_1+m_1\sqrt{a}\right)\right.\\ &\qquad\left.+\frac{\sqrt{a}(x-\bar{x})}{2}+\frac{\sqrt{a}\,e^{\sqrt{a}c_1}}{1-e^{\sqrt{a}c_1}}u\right\},\end{aligned}$$

where γ_1 and γ_1' are constants.

Similarly we have

$$\begin{aligned}\psi_2(v) &= \gamma_2' + \gamma_2 v - \frac{2}{\epsilon}\log\left\{\sqrt{ac}\coth\frac{\sqrt{a}(x-\bar{x})}{2}\coth\frac{\sqrt{c}(y-\bar{y})}{2}-\epsilon_1\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2}\right.\\ &\qquad\left.-\epsilon_2\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2}+\epsilon_1\epsilon_2+2m_1m_2\epsilon\delta\right\}\\ &\qquad-\frac{2}{\epsilon}\left\{\log\left(\frac{\sinh\frac{\sqrt{c}(y-\bar{y})}{2}}{\sqrt{c}}\right)-\log\left(\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2}+n_2+m_2\sqrt{c}\right)\right.\\ &\qquad\left.+\frac{\sqrt{c}(y-\bar{y})}{2}+\frac{\sqrt{c}\,e^{\sqrt{c}c_2}}{1-e^{\sqrt{c}c_2}}v\right\},\end{aligned}$$

where γ_2 and γ_2' are constants.

Therefore

$$\psi_1(u)-\psi_2(v)=\phi_2(x)-\phi_1(y)$$

$$=\gamma_1'-\gamma_2'+\gamma_1 u-\gamma_2 v-\frac{2}{\epsilon}\left\{\log\left(\frac{\sinh\frac{\sqrt{a}\,(x-\bar{x})}{2}}{\sqrt{a}}\right)-\log\left(\frac{\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}}{\sqrt{c}}\right)\right\}$$

$$+\frac{2}{\epsilon}\left\{\log\left(\sqrt{c}\coth\frac{\sqrt{c}(y-\bar{y})}{2}+n_1+m_1\sqrt{a}\right)-\log\left(\sqrt{a}\coth\frac{\sqrt{a}\,(x-\bar{x})}{2}+n_2+m_2\sqrt{c}\right)\right\}$$

$$-\frac{2}{\epsilon}\left\{\frac{\sqrt{a}\,e^{\sqrt{a}c_1}}{1-e^{\sqrt{a}c_1}}u-\frac{\sqrt{c}\,e^{\sqrt{c}c_2}}{1-e^{\sqrt{c}c_2}}v+\frac{\sqrt{a}\,(x-\bar{x})}{2}-\frac{\sqrt{c}\,(y-\bar{y})}{2}\right\},$$

from which it follows that

$$\phi_1(y)=\gamma-\gamma_1'+\frac{2\gamma_1}{\sqrt{a}}\coth^{-1}\left(\frac{\sqrt{c}}{m_1\sqrt{a}}\coth\frac{\sqrt{c}\,(y-\bar{y})}{2}+\frac{n_1}{m_1\sqrt{a}}\right)$$

$$+\gamma_2(y-\bar{y})-\frac{2}{\epsilon}\log\frac{\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}}{\sqrt{c}}-\frac{\epsilon_2}{\epsilon}(y-\bar{y})$$

$$-\frac{2}{\epsilon}\log\left(\sqrt{c}\coth\frac{\sqrt{c}\,(y-\bar{y})}{2}+n_1+m_1\sqrt{a}\right)$$

$$-\frac{2(\epsilon_1-\sqrt{a})}{\epsilon\sqrt{a}}\coth^{-1}\left(\frac{\sqrt{c}}{m_1\sqrt{a}}\coth\frac{\sqrt{c}\,(y-\bar{y})}{2}+\frac{n_1}{m_1\sqrt{a}}\right),$$

$$\phi_2(x)=\gamma-\gamma_2'+\gamma_1(x-\bar{x})+\frac{2\gamma_2}{\sqrt{c}}\coth^{-1}\left(\frac{\sqrt{a}}{m_2\sqrt{c}}\coth\frac{\sqrt{a}\,(x-\bar{x})}{2}+\frac{n_2}{m_2\sqrt{c}}\right)$$

$$-\frac{2}{\epsilon}\log\frac{\sinh\frac{\sqrt{a}\,(x-\bar{x})}{2}}{\sqrt{a}}-\frac{\epsilon_1}{\epsilon}(x-\bar{x})$$

$$-\frac{2}{\epsilon}\log\left(\sqrt{a}\coth\frac{\sqrt{a}(x-\bar{x})}{2}+n_2+m_2\sqrt{c}\right)$$

$$-\frac{2(\epsilon_2-\sqrt{c})}{\epsilon\sqrt{c}}\coth^{-1}\left(\frac{\sqrt{a}}{m_2\sqrt{c}}\coth\frac{\sqrt{a}\,(x-\bar{x})}{2}+\frac{n_2}{m_2\sqrt{c}}\right),$$

where γ is a constant. We have now, from Equation (i),

$$\log z=\phi_1(y)+\psi_1(u)=\phi_2(x)+\psi_2(v)$$

$$=\gamma+\gamma_1(x-\bar{x})+\gamma_2(y-\bar{y})$$

$$-\frac{2}{\epsilon}\left\{\frac{\epsilon_1}{2}(x-\bar{x})+\frac{\epsilon_2}{2}(y-\bar{y})+\log\frac{\sinh\frac{\sqrt{a}\,(x-\bar{x})}{2}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}}{\sqrt{ac}}\right\}$$

$$-\frac{2}{\epsilon}\log\left\{2m_1m_2\epsilon\delta+\left(\sqrt{a}\coth\frac{\sqrt{a}\,(x-\bar{x})}{2}-\epsilon_1\right)\left(\sqrt{c}\coth\frac{\sqrt{c}\,(y-\bar{y})}{2}-\epsilon_2\right)\right\}$$

$$=\gamma+\left(\gamma_1-\frac{\epsilon_1}{\epsilon}\right)(x-\bar{x})+\left(\gamma_2-\frac{\epsilon_2}{\epsilon}\right)(y-\bar{y})$$

$$-\frac{2}{\epsilon}\log\left\{\left(\cosh\frac{\sqrt{a}(x-\bar{x})}{2}-\frac{\epsilon_1}{\sqrt{a}}\sinh\frac{\sqrt{a}(x-\bar{x})}{2}\right)\left(\cosh\frac{\sqrt{c}(y-\bar{y})}{2}\right.\right.$$

$$\left.\left.-\frac{\epsilon_2}{\sqrt{c}}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right)+\frac{2m_1m_2\epsilon\delta}{\sqrt{ac}}\sinh\frac{\sqrt{a}\,(x-\bar{x})}{2}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right\},$$

that is,

$$z = e^{\gamma+\left(\gamma_1-\frac{\epsilon_1}{\epsilon}\right)(x-\bar{x})+\left(\gamma_2-\frac{\epsilon_2}{\epsilon}\right)(y-\bar{y})}$$
$$\times\left\{\left(\cosh\frac{\sqrt{a}\,(x-\bar{x})}{2}-\frac{\epsilon_1}{\sqrt{a}}\sinh\frac{\sqrt{a}\,(x-\bar{x})}{2}\right)\left(\cosh\frac{\sqrt{c}\,(y-\bar{y})}{2}-\frac{\epsilon_2}{\sqrt{c}}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right)\right.$$
$$\left.+\frac{2m_1m_2\epsilon\delta}{\sqrt{ac}}\sinh\frac{\sqrt{a}\,(x-\bar{x})}{2}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right\}^{-\frac{2}{\epsilon}};$$

and the equations to the regression curves may be written as

$$m_1\sqrt{a}\coth\frac{\sqrt{a}\,(f_1(y)-\bar{x})}{2}=\sqrt{c}\coth\frac{\sqrt{c}\,(y-\bar{y})}{2}+n_1,$$

$$m_2\sqrt{c}\coth\frac{\sqrt{c}\,(f_2(x)-\bar{y})}{2}=\sqrt{a}\coth\frac{\sqrt{a}\,(x-\bar{x})}{2}+n_2.$$

These forms are probably the most convenient, because, for the following cases, the equations to the regression curves are easily deduced from them.

Case (ii). $a=0,\quad c>0.$

In this case we have the equations to the regression lines

$$\frac{2m_1}{f_1(y)-\bar{x}}=\sqrt{c}\coth\frac{\sqrt{c}\,(y-\bar{y})}{2}+n_1,$$

$$m_2\sqrt{c}\coth\frac{\sqrt{c}\,(f_2(x)-\bar{y})}{2}=\frac{2}{x-\bar{x}}+n_2;$$

and the equation to the surface

$$z=e^{\gamma+\left(\gamma_1-\frac{\epsilon_1}{\epsilon}\right)(x-\bar{x})+\left(\gamma_2-\frac{\epsilon_2}{\epsilon}\right)(y-\bar{y})}$$
$$\times\left\{\left(1-\frac{\epsilon_1(x-\bar{x})}{2}\right)\left(\cosh\frac{\sqrt{c}\,(y-\bar{y})}{2}-\frac{\epsilon_2}{\sqrt{c}}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right)\right.$$
$$\left.+\frac{m_1m_2\epsilon\delta}{\sqrt{c}}(x-\bar{x})\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right\}^{-\frac{2}{\epsilon}}.$$

Case (iii). $a<0,\quad c>0.$

In this case we have the equations to the curves of regression

$$m_1\sqrt{-a}\cot\frac{\sqrt{-a}\,(f_1(y)-\bar{x})}{2}=\sqrt{c}\coth\frac{\sqrt{c}\,(y-\bar{y})}{2}+n_1,$$

$$m_2\sqrt{c}\coth\frac{\sqrt{c}\,(f_2(x)-\bar{y})}{2}=\sqrt{-a}\cot\frac{\sqrt{-a}\,(x-\bar{x})}{2}+n_2;$$

and the equation to the surface

$$z=e^{\gamma+\left(\gamma_1-\frac{\epsilon_1}{\epsilon}\right)(x-\bar{x})+\left(\gamma_2-\frac{\epsilon_2}{\epsilon}\right)(y-\bar{y})}$$
$$\times\left\{\left(\cos\frac{\sqrt{-a}\,(x-\bar{x})}{2}-\frac{\epsilon_1}{\sqrt{-a}}\sin\frac{\sqrt{-a}\,(x-\bar{x})}{2}\right)\left(\cosh\frac{\sqrt{c}\,(y-\bar{y})}{2}-\frac{\epsilon_2}{\sqrt{c}}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right)\right.$$
$$\left.+\frac{2m_1m_2\epsilon\delta}{\sqrt{-ac}}\sin\frac{\sqrt{-a}\,(x-\bar{x})}{2}\sinh\frac{\sqrt{c}\,(y-\bar{y})}{2}\right\}^{-\frac{2}{\epsilon}}.$$

Case (iv). $a=0, \quad c=0.$

In this case we have the equations to the curves of regression

$$\frac{2m_1}{f_1(y)-\bar{x}}=\frac{2}{y-\bar{y}}+n_1,$$

$$\frac{2m_2}{f_2(x)-\bar{y}}=\frac{2}{x-\bar{x}}+n_2;$$

and the equation to the surface

$$z=e^{\gamma+\left(\gamma_1-\frac{\epsilon_1}{\epsilon}\right)(x-\bar{x})+\left(\gamma_2-\frac{\epsilon_2}{\epsilon}\right)(y-\bar{y})}$$
$$\times\left\{\left(1-\frac{\epsilon_1(x-\bar{x})}{2}\right)\left(1-\frac{\epsilon_2(y-\bar{y})}{2}\right)+\frac{m_1m_2\epsilon\delta}{2}(x-\bar{x})(y-\bar{y})\right\}^{-\frac{2}{\epsilon}}.$$

The arrays are here both Pearson curves of Type III.

Case (v). $a<0, \quad c=0.$

In this case we have the equations to the regression lines

$$m_1\sqrt{-a}\cot\frac{\sqrt{-a}\,(f_1(y)-\bar{x})}{2}=\frac{2}{y-\bar{y}}+n_1,$$

$$\frac{2m_2}{f_2(x)-\bar{y}}=\sqrt{-a}\cot\frac{\sqrt{-a}\,(x-\bar{x})}{2}+n_2;$$

and the equation to the surface

$$z=e^{\gamma+\left(\gamma_1-\frac{\epsilon_1}{\epsilon}\right)(x-\bar{x})+\left(\gamma_2-\frac{\epsilon_2}{\epsilon}\right)(y-\bar{y})}$$
$$\times\left\{\left(\cos\frac{\sqrt{-a}\,(x-\bar{x})}{2}-\frac{\epsilon_1}{\sqrt{-a}}\sin\frac{\sqrt{-a}\,(x-\bar{x})}{2}\right)\left(1-\frac{\epsilon_2(y-\bar{y})}{2}\right)\right.$$
$$\left.+\frac{m_1m_2\epsilon\delta}{\sqrt{-a}}\sin\frac{\sqrt{-a}\,(x-\bar{x})}{2}\cdot(y-\bar{y})\right\}^{-\frac{2}{\epsilon}}.$$

Case (vi). $a<0, \quad c<0.$

In this case we have the equations to the regression curves

$$m_1\sqrt{-a}\cot\frac{\sqrt{-a}\,(f_1(y)-\bar{x})}{2}=\sqrt{-c}\cot\frac{\sqrt{-c}\,(y-\bar{y})}{2}+n_1,$$

$$m_2\sqrt{-c}\cot\frac{\sqrt{-c}\,(f_2(x)-\bar{y})}{2}=\sqrt{-a}\cot\frac{\sqrt{-a}\,(x-\bar{x})}{2}+n_2;$$

and the equation to the surface

$$z=e^{\gamma+\left(\gamma_1-\frac{\epsilon_1}{\epsilon}\right)(x-\bar{x})+\left(\gamma_2-\frac{\epsilon_2}{\epsilon}\right)(y-\bar{y})}$$
$$\times\left\{\left(\cos\frac{\sqrt{-a}\,(x-\bar{x})}{2}-\frac{\epsilon_1}{\sqrt{-a}}\sin\frac{\sqrt{-a}\,(x-\bar{x})}{2}\right)\left(\cos\frac{\sqrt{-c}\,(y-\bar{y})}{2}-\frac{\epsilon_2}{\sqrt{-c}}\sin\frac{\sqrt{-c}\,(y-\bar{y})}{2}\right)\right.$$
$$\left.+\frac{2m_1m_2\epsilon\delta}{\sqrt{ac}}\sin\frac{\sqrt{-a}\,(x-\bar{x})}{2}\sin\frac{\sqrt{-c}\,(y-\bar{y})}{2}\right\}^{-\frac{2}{\epsilon}}$$

Case (vii). $\epsilon = 0.$

In this case it may be anticipated that we shall reach the result of Part I of my paper. We have already found the relations

$$a = \epsilon_1^2 - 2m_2\epsilon\delta, \quad \text{and} \quad c = \epsilon_2^2 - 2m_1\epsilon\delta.$$

Hence, using the relations

$$\epsilon_1 = \sqrt{a} + \frac{2m_2\epsilon\delta}{\epsilon_1 + \sqrt{a}}, \quad \epsilon_2 = \sqrt{c} + \frac{2m_1\epsilon\delta}{\epsilon_2 + \sqrt{c}},$$

the equation to the surface will be rewritten in a form

$$z = e^{\gamma + \left(\gamma_1 - \frac{2m_2\delta}{\epsilon_1+\sqrt{a}}\right)(x-\bar{x}) + \left(\gamma_2 - \frac{2m_1\delta}{\epsilon_2+\sqrt{c}}\right)(y-\bar{y})}$$
$$\times \left[\left\{1 - \frac{m_2\epsilon\delta}{\sqrt{a}\,(\epsilon_1+\sqrt{a})}\,(e^{\sqrt{a}\,(x-\bar{x})} - 1)\right\}\left\{1 - \frac{m_1\epsilon\delta}{\sqrt{c}\,(\epsilon_2+\sqrt{c})}\,(e^{\sqrt{c}\,(y-\bar{y})} - 1)\right\}\right.$$
$$\left. + \frac{m_1 m_2\epsilon\delta}{2\sqrt{ac}}\,(e^{\sqrt{a}\,(x-\bar{x})} - 1)\,(e^{\sqrt{c}\,(y-\bar{y})} - 1)\right]^{-\frac{2}{\epsilon}}.$$

Therefore, making $\epsilon = 0$, we have

$$z = e^{\gamma + \left(\gamma_1 - \frac{m_2\delta}{\sqrt{a}}\right)(x-\bar{x}) + \left(\gamma_2 - \frac{m_1\delta}{\sqrt{c}}\right)(y-\bar{y})}$$
$$\times e^{\delta\left\{\frac{m_2}{a}\,(e^{\sqrt{a}(x-\bar{x})} - 1) + \frac{m_1}{c}\,(e^{\sqrt{c}(y-\bar{y}} - 1) - \frac{m_1 m_2}{\sqrt{ac}}\,(e^{\sqrt{a}(x-\bar{x})} - 1)(e^{\sqrt{c}(y-\bar{y})} - 1)\right\}},$$

which is the same form as was found in Part I (p. 84) of my paper. The equations to the lines of regression now become

$$m_1\sqrt{a}\left\{\coth\frac{\sqrt{a}\,(f_1(y) - \bar{x})}{2} + 1\right\} = \sqrt{c}\left\{\coth\frac{\sqrt{c}\,(y-\bar{y})}{2} - 1\right\},$$

$$m_2\sqrt{c}\left\{\coth\frac{\sqrt{c}\,(f_2(x) - \bar{y})}{2} + 1\right\} = \sqrt{a}\left\{\coth\frac{\sqrt{a}(x-\bar{x})}{2} - 1\right\}.$$

(5) Let us consider the case in which the functional equation becomes

$$\log z = \phi_1(y) + \psi_1\{(y+g_1)x + (f_1y + c_1)\} = \phi_2(x) + \psi_2\{(x+f_2)y + (g_2x + c_2)\},$$

where g_1, f_1, c_1, g_2, f_2, and c_2 are all constants.

This case corresponds to that in which the regression curves are equilateral hyperbolae both ways, and the standard deviations are in the form $\frac{1}{y+g_1}$ and $\frac{1}{x+f_2}$ respectively.

Differentiate with regard to x

$$(y+g_1)\,\psi_1'\{(y+g_1)x + (f_1y + c_1)\} = \phi_2'(x) + (y+g_2)\,\psi_2'\{(x+f_2)y + (g_2x + c_2)\},$$

and then with regard to y, and we find

$$\psi_1'(u) + (u + f_1g_1 - c_1)\,\psi_1''(u) = \psi_2'(v) + (v + f_2g_2 - c_2)\,\psi_2''(v),$$

where $$u = xy + f_1y + g_1x + c_1$$

and $$v = xy + f_2y + g_2x + c_2.$$

In the above equation, the left-hand side is a function of u only and the right-hand side a function of v only; therefore they must be equal to a constant, γ say. Hence we have

$$\psi_1'(u) + (u + f_1 g_1 - c_1)\,\psi_1''(u) = \gamma,$$
$$\psi_2'(v) + (v + f_2 g_2 - c_2)\,\psi_2''(v) = \gamma.$$

By integration we easily find that

$$\psi_1(u) = \gamma u + \gamma_1 \log(u + f_1 g_1 - c_1),$$
$$\psi_2(v) = \gamma v + \gamma_2 \log(v + f_2 g_2 - c_2),$$

where γ_1 and γ_2 are constants.

Replacing u and v by their values in x and y we have

$$\psi_1(u) = \gamma(xy + f_1 y + g_1 x + c_1) + \gamma_1 \log(x + f_1)(y + g_1),$$
$$\psi_2(v) = \gamma(xy + f_2 y + g_2 x + c_2) + \gamma_2 \log(x + f_2)(y + g_2).$$

Thus we reach

$$\begin{aligned}\psi_1(u) - \psi_2(v) &= \gamma\{(g_1 - g_2)x + (f_1 - f_2)y + c_1 - c_2\} + \gamma_1 \log(x + f_1) - \gamma_2 \log(x + f_2) \\ &\qquad + \gamma_1 \log(y + g_1) - \gamma_2 \log(y + g_2) \\ &= \phi_2(x) - \phi_1(y).\end{aligned}$$

Hence

$$\phi_2(x) = \delta + \gamma\{(g_1 - g_2)x - c_2\} + \gamma_1 \log(x + f_1) - \gamma_2 \log(x + f_2),$$
$$\phi_1(y) = \delta + \gamma\{(f_2 - f_1)y - c_1\} - \gamma_1 \log(y + g_1) + \gamma_2 \log(y + g_2);$$

where δ is a constant.

Now since we have

$$\begin{aligned}\log z &= \phi_1(y) + \psi_1(u) = \phi_2(x) + \psi_2(v) \\ &= \delta + \gamma(xy + g_1 x + f_2 y) + \gamma_1 \log(x + f_1) + \gamma_2 \log(y + g_2) \\ &= \epsilon + \gamma(x + f_2)(y + g_1) + \gamma_1 \log(x + f_1) + \gamma_2 \log(y + g_2);\end{aligned}$$

where ϵ is a constant; consequently we have

$$z = z_0 (x + f_1)^{\gamma_1} (y + g_2)^{\gamma_2} e^{\gamma (x + f_2)(y + g_1)},$$

which gives arrays both ways as Pearson's Type III curves.

(6) Let us next suppose that the regression lines are both linear and that the standard deviations are represented by $\sqrt{y^2 + c_1}$ and $\sqrt{x^2 + c_2}$ respectively, where c_1 and c_2 are non-vanishing constants. Then the functional equation of the heteroscedasticity will become

$$\log z = \phi_1(y) + \psi_1\left(\frac{x - m_1 y}{\sqrt{y^2 + c_1}}\right) = \phi_2(x) + \psi_2\left(\frac{y - m_2 x}{\sqrt{x^2 + c_2}}\right), \quad c_1 c_2 \neq 0;$$

where the mean of the whole population is the origin and the regression lines are

$$\tilde{x} = m_1 y \quad \text{and} \quad \tilde{y} = m_2 x.$$

Differentiate this equation with regard to x and y, and we find:

$$\frac{1}{\sqrt{y^2+c_1}}\psi_1'\left(\frac{x-m_1y}{\sqrt{y^2+c_1}}\right)=\phi_2'(x)-\frac{xy+m_2c_2}{\sqrt{(x^2+c_2)^3}}\psi_2'\left(\frac{y-m_2x}{\sqrt{x^2+c_2}}\right),$$

and

$$\phi_1'(y)-\frac{xy+m_1c_1}{\sqrt{(y^2+c_1)^3}}\psi_1'\left(\frac{x-m_1y}{\sqrt{y^2+c_1}}\right)=\frac{1}{\sqrt{x^2+c_2}}\psi_2'\left(\frac{y-m_2x}{\sqrt{x^2+c_2}}\right).$$

Solving these equations we have

$$\frac{\psi_1'(u)}{\sqrt{(y^2+c_1)^3}}=\frac{(x^2+c_2)\phi_2'(x)-(xy+m_2c_2)\phi_1'(y)}{c_1x^2-(m_1c_1+m_2c_2)xy+c_2y^2+(1-m_1m_2)c_1c_2} \quad \text{.....(xiv),}$$

$$\frac{\psi_2'(v)}{\sqrt{(x^2+c_2)^3}}=\frac{(y^2+c_1)\phi_1'(y)-(xy+m_1c_1)\phi_2'(x)}{c_1x^2-(m_1c_1+m_2c_2)xy+c_2y^2+(1-m_1m_2)c_1c_2} \quad \text{...(xiv) } bis,$$

where

$$u=\frac{x-m_1y}{\sqrt{y^2+c_1}} \quad \text{and} \quad v=\frac{y-m_2x}{\sqrt{x^2+c_2}}.$$

Now in order to eliminate the function $\psi_1'(u)$, let us take the logarithmic differential of Equation (xiv) with regard to x and y,

$$\frac{1}{\sqrt{y^2+c_1}}\frac{\psi_1''(u)}{\psi_1'(u)}=\frac{(x^2+c_2)\phi_2''(x)+2x\phi_2'(x)-y\phi_1'(y)}{(x^2+c_2)\phi_2'(x)-(xy+m_2c_2)\phi_1'(y)}-\frac{2c_1x-(m_1c_1+m_2c_2)y}{c_1x^2-(m_1c_1+m_2c_2)xy+c_2y^2+(1-m_1m_2)c_1c_2},$$

$$-\frac{3y}{y^2+c_1}-\frac{xy+m_1c_1}{\sqrt{(y^2+c_1)^3}}\frac{\psi_1''(u)}{\psi_1'(u)}=\frac{-(xy+m_2c_2)\phi_1''(y)-x\phi_1'(y)}{(x^2+c_2)\phi_2'(x)-(xy+m_2c_2)\phi_1'(y)}-\frac{2c_2y-(m_1c_1+m_2c_2)x}{c_1x^2-(m_1c_1+m_2c_2)xy+c_2y^2+(1-m_1m_2)c_1c_2}.$$

Multiplying $xy+m_1c_1$ into the former, y^2+c_1 into the latter and adding together we have, after some reductions,

$$\begin{aligned}&\{(xy+m_2c_2)(y^2+c_1)\phi_1''(y)+2(xy+m_2c_2)y\phi_1'(y)+x(y^2+c_1)\phi_1'(y)\}\\&\qquad\times\{c_1x^2-(m_1c_1+m_2c_2)xy+c_2y^2+(1-m_1m_2)c_1c_2\}\\&\quad+c_2(m_1c_1-m_2c_2)(y-m_2x)(y^2+c_1)\phi_1'(y)\\&=\{(xy+m_1c_1)(x^2+c_2)\phi_2''(x)+2(xy+m_1c_1)x\phi_2'(x)+y(x^2+c_2)\phi_2'(x)\}\\&\qquad\times\{c_1x^2-(m_1c_1+m_2c_2)xy+c_2y^2+(1-m_1m_2)c_1c_2\}\\&\quad+c_1(m_2c_2-m_1c_1)(x-m_1y)(x^2+c_2)\phi_2'(x).\end{aligned}$$

For simplicity let us put $(y^2+c_1)\phi_1'(y)=\chi_1(y)$ and $(x^2+c_2)\phi_2'(x)=\chi_2(x)$, and arrange in powers of x on the left-hand side, in powers of y on the right-hand side, we have

$$\begin{aligned}&c_1(y\chi_1'(y)+\chi_1(y))x^2+\{m_2c_1c_2\chi_1'(y)-(m_1c_1+m_2c_2)y(y\chi_1'(y)+\chi_1(y))\}x^2\\&\quad+\{[(1-2m_1m_2)c_1c_2+c_2y^2](y\chi_1'(y)+\chi_1(y))-m_2^2c_2^2(y\chi_1'(y)-\chi_1(y))\}x\\&\quad+\{m_2c_2[(1-m_1m_2)c_1c_2+c_2y^2]\chi_1'(y)+c_2(m_1c_1-m_2c_2)y\chi_1(y)\}\\&=c_2(x\chi_2'(x)+\chi_2(x))y^2+\{m_1c_1c_2\chi_2'(x)-(m_1c_1+m_2c_2)x(x\chi_2'(x)+\chi_2(x))\}y^2\\&\quad+\{[(1-2m_1m_2)c_1c_2+c_1x^2](x\chi_2'(x)+\chi_2(x))-m_1^2c_1^2(x\chi_2'(x)-\chi_2(x))\}y\\&\quad+\{m_1c_1[(1-m_1m_2)c_1c_2+c_1x^2]\chi_2'(x)+c_1(m_2c_2-m_1c_1)x\chi_2(x)\}\quad\text{.....(xv).}\end{aligned}$$

The left-hand side is clearly a cubic in x and the right-hand side a cubic in y; therefore, on the left-hand side, the coefficient of every power of x must be cubic in y and, on the right-hand side, the coefficient of every power of y must be cubic in x.

Now, since $c_1 \neq 0$, let us assume that

$$y\chi_1'(y)+\chi_1(y)=A_1+2A_2y+3A_3y^2+4A_4y^3,$$

where A_1, A_2, A_3 and A_4 are constants to be determined.

By integration we have

$$y\chi_1(y)=A_0+A_1y+A_2y^2+A_3y^3+A_4y^4,$$

where A_0 is another constant. And further we have

$$y\chi_1'(y)=-\frac{A_0}{y}+A_2y+2A_3y^2+3A_4y^3.$$

Therefore the coefficient of x in the left-hand side of Equation (xv) now becomes

$$m_2^2c_2^2\left(\frac{2A_0}{y}+A_1-A_3y^2-2A_4y^3\right)+\{(1-2m_1m_2)c_1c_2+c_2y^2\}(A_1+2A_2y+3A_3y^2+4A_4y^3),$$

which must be cubic in y; hence it follows that

$$A_0=A_3=A_4=0.$$

The left-hand side of Equation (xv) now becomes

$$\begin{aligned}&c_1(A_1+2A_2y)x^2+\{m_2c_1c_2A_2-(m_1c_1+m_2c_2)A_1y-2(m_1c_1+m_2c_2)A_2y^2\}x^2\\&\quad+\{m_2^2c_2^2A_1+(1-2m_1m_2)c_1c_2A_1+2(1-2m_1m_2)c_1c_2A_2y+c_2A_1y^2+2c_2A_2y^3\}x\\&\quad+\{(1-m_1m_2)m_2c_1c_2^2A_2+c_2(m_1c_1-m_2c_2)A_1y+m_1c_1c_2A_2y^2\}\\&=2c_2A_2xy^3+\{m_1c_1c_2A_2+c_2A_1x-2(m_1c_1+m_2c_2)A_2x^2\}y^2\\&\quad+\{c_2(m_1c_1-m_2c_2)A_1+2(1-2m_1m_2)c_1c_2A_2x-(m_1c_1+m_2c_2)A_1x^2+2c_1A_2x^3\}y\\&\quad+\{(1-m_1m_2)m_2c_1c_2^2A_2+[(1-2m_1m_2)c_1c_2A_1+m_2^2c_2^2A_1]x+m_2c_1c_2A_2x^2+c_1A_1x^3\},\end{aligned}$$

which must be identically equal to the right-hand side of Equation (xv).

First of all we have

$$x\chi_2'(x)+\chi_2(x)=2A_2x,$$

or by integration

$$x\chi_2(x)=A+A_2x^2,$$

where A is another constant. And the coefficient of y on the right-hand side of Equation (xv) becomes

$$m_1^2c_1^2\frac{A}{x}+2(1-2m_1m_2)c_1c_2A_2x+2c_1A_2x^2,$$

which ought to be equal to

$$c_2(m_1c_1-m_2c_2)A_1+2(1-2m_1m_2)c_1c_2A_2x-(m_1c_1+m_2c_2)A_1x^2+2c_1A_2x^3.$$

Therefore, by virtue of $c_1c_2 \neq 0$, we have

$$A=A_1=0.$$

Finally, comparing the constant terms, we have

$$(1 - m_1 m_2)(m_1 c_1 - m_2 c_2) A_2 = 0.$$

When $A_2 = 0$, we have $\chi_2(x) = \chi_1(y) = 0$,

and consequently $\phi_1'(y) = \phi_2'(x) = \psi_1'(u) = \psi_2'(v) = 0$,

which is impossible unless z itself is a constant.

(i) When $m_1 c_1 - m_2 c_2 = 0$,

we have $m_1 c_1 = m_2 c_2 = m_1 m_2 c$, say.

We have also $\phi_1'(y) = \dfrac{2A_2 y}{y^2 + m_2 c}$, $\phi_2'(x) = \dfrac{2A_2 x}{x^2 + m_1 c}$;

and, by integration

$$\phi_1(y) = A_2 \log(y^2 + m_2 c) + \text{const.}, \quad \phi_2(x) = A_2 \log(x^2 + m_1 c) + \text{const.}$$

Further, from Equations (xiv) and (xiv) *bis*, we have

$$\psi_1'(u) = \frac{2m_2 A_2 u}{m_2 u^2 + m_1(1 - m_1 m_2)}, \quad \psi_2'(v) = \frac{2m_1 A_2 v}{m_1 v^2 + m_2(1 - m_1 m_2)};$$

integrating these relations,

$$\psi_1(u) = A_2 \log(m_2 u^2 + m_1(1 - m_1 m_2)) + \text{const.},$$
$$\psi_2(v) = A_2 \log(m_1 v^2 + m_2(1 - m_1 m_2)) + \text{const.}$$

Therefore, we have

$$\log z = \phi_1(y) + \psi_1(u) = \phi_2(x) + \psi_2(v)$$
$$= A_2 \log\{m_2 x^2 - 2m_1 m_2 xy + m_1 y^2 + m_1 m_2(1 - m_1 m_2) c\} + \text{const.},$$

or

$$z = z_0 \left(1 + \frac{\dfrac{x^2}{m_1} - 2xy + \dfrac{y^2}{m_2}}{(1 - m_1 m_2) c}\right)^{A_2}$$

This will become the usual form*

$$z = z_0 \left(1 + \frac{\dfrac{x^2}{\sigma_1^2} - \dfrac{2rxy}{\sigma_1 \sigma_2} + \dfrac{y^2}{\sigma_2^2}}{2\lambda(1 - r^2)}\right)^{-\lambda - 2},$$

when

$$m_1 = \frac{r\sigma_1}{\sigma_2}, \quad m_2 = \frac{r\sigma_2}{\sigma_1}, \quad c = \frac{2\lambda \sigma_1 \sigma_2}{r}, \quad A_2 = -\lambda - 2.$$

(ii) When $m_1 m_2 = 1$,

we have

$$\phi_1(y) = A_2 \log(y^2 + c_1) + \text{const.}, \quad \phi_2(x) = A_2 \log(x^2 + c_2) + \text{const.};$$

and $\psi_1(u) = 2A_2 \log u + \text{const.}$, $\psi_2(v) = 2A_2 \log v + \text{const.}$

* First discussed with methods of fitting by Pearson at the Meeting of the Society of Biometricians and Mathematical Statisticians on May 15, 1922.

Therefore

$$\log z = A_3 \log (m_2 x^2 - 2xy + m_1 y^2) + \text{const.},$$

or

$$z = z_0 (m_2 x^2 - 2xy + m_1 y^2)^{A_3} = z_0 (\sqrt{m_2}\, x - \sqrt{m_1}\, y)^{2A_3}$$

which is a cylindrical surface having an infinite volume.

I at first tried to find the surface satisfying the functional equation

$$\phi_1(y) + \psi_1\left(\frac{x - m_1 y}{\sqrt{y^2 + 2b_1 y + c_1}}\right) = \phi_2(x) + \psi_2\left(\frac{y - m_2 x}{\sqrt{x^2 + 2b_2 x + c_2}}\right).$$

The analysis in this case was much more complicated than the above, but by a similar method I found that the only possible form was also

$$z = z_0 \left(1 + \frac{\dfrac{x^2}{m_1} - 2xy + \dfrac{y^2}{m_2}}{(1 - m_1 m_2)\, c}\right)^{A_3}$$

(To be continued)

NOTES ON SKEW FREQUENCY SURFACES.

By KARL PEARSON, F.R.S.

(1) *Historical.* In 1895 after the publication of my memoir on Skew Variation* I proceeded to the problem of skew frequency surfaces, and gave much time to attempting its solution. By actually analysing skew correlation data, I was able to convince myself that for skew correlation surfaces in general there might be principal inertial axes of the contour system, but if these existed, they were not directions of independent probability†. That is to say that if the frequency surface were $z = \phi(x, y)$, it in most cases could not be thrown into the form $z = f_1(x') \times f_2(y')$, by a rotation of axes. This cleared out of the way a good many of the *a priori* possibilities, e.g. that a generalised surface could be obtained by taking the product of two of my skew frequency curves and rotating the axes. I did indeed find that we could not do as we do with the normal surface, take the product of two normal curves, and rotate the axes to get correlated variates. This conclusion was reached not theoretically, but by analysing by contour lines one or two markedly skew surfaces.

As a next step, dropping axes of independent probability, I endeavoured to determine a surface in two dimensions which in the sense of my memoir on Skew Variation should be a "parallel" to the double hypergeometrical series as my curves were "parallels" to the single hypergeometrical series. I obtained the differential equations to such series, and they led up to two very interesting points: (i) the regression lines were both cubics and (ii) their intersection was *not* the mean of the bi-variate surface. But I failed entirely to integrate my equations. I have returned to them again and again during nearly thirty years with no greater success. They were only integrable, when rather arbitrary relations were imposed on the constants. In these cases I could get solutions generally involving—as might be anticipated—my skew curves as array curves, but not always as marginal totals.

I did not, however, trust finally to my own unaided efforts, but put my differential equations before fresher and younger minds.

In 1901, when Professor L. N. G. Filon was an assistant in my laboratory, I placed the problem before him. He obtained three or four surfaces and extended some of them to multivariate surfaces of which I will say a few words later. He considered the fitting of such surfaces by the method of moments and product moments. But in all these cases the correlation could be shown to be a function of

* *Phil. Trans.* Vol. 186 A, pp. 343—414, 1895.

† Schols and others have dealt with the problem of independent axes of probability, but do not seem to have inquired experimentally as to how far such axes actually exist.

the variation coefficients, i.e. if β_1, β_2 and β_1', β_2' are the β's of the two correlated variates, then not only is there a relation between these β's, but the correlation coefficient is a function of them. The normal surface shows us that given certain values of the β's (i.e. $\beta_1 = \beta_1' = 0$, $\beta_2 = \beta_2' = 3$), then the correlation will be independent of them. Some time ago I was able to find a correlation surface subject to the conditions $\beta_1 = \beta_1' = 0$, $\beta_2 = \beta_2' =$ an arbitrary value, for which the correlation coefficient is free, i.e. can take any value. Professor Seimatsu Narumi has recently shown that my surface is the only possible one in which for both variates linearity of regression is combined with parabolic variance* of the arrays. As far as I am aware this is the only surface yet considered, in which while the β's are restricted, r, the correlation coefficient, is certainly free. I feel this point is a very important one, because one is hardly prepared at present to accept a relationship for skew surfaces between variation and correlation. On this ground I suggested to Dr Filon to hold back his investigation until a more general system of surfaces could be reached.

In 1913 and 1914 Dr L. Isserlis was working in my laboratory and I put my differential equations before him. Again a general solution was not reached, but Dr Isserlis succeeded in fitting data with a double hypergeometrical series—a useful problem solved—and he also reached again one of Professor Filon's types, and showed in his own way how to fit it. I again suggested (i) that what we need is a general system of surfaces, and (ii) that a particular solution in which the correlation is of necessity a function of the variation is not wholly satisfactory. He also agreed to suspend publication. In 1919 I once more placed my differential equations before Mr E. C. Rhodes who had come as an assistant to the Biometric Laboratory. Again fate was against us, no general solution was obtained, but like Dr Isserlis and Dr Filon, Mr Rhodes obtained a special solution. This solution was rather more general than one of Dr Filon's types, although by a rotation of the axes it is convertible into one of these; such rotation, however, modifies the correlated variables. Mr Rhodes' equation† leads to very complicated expressions for the marginal totals, and I do not feel able to state definitely whether the correlation is or is not a function of the marginal β's. I am inclined from his results to believe that it is‡. There is further a restriction, I think, on his variation constants the marginal β's, but it is not easy to determine its exact nature. I was not very happy about Mr Rhodes' surface with its complicated marginal total curves and regression curves; it seemed to make a breach between previous methods of dealing with variation and regression, and suggested that these must be approached from an entirely different aspect. At the same time, with a larger number of constants, it departed from the Filon and Isserlis types, and might possibly give an independence of variation and correlation. For, after all, it is rather a shock to suggest that from the frequency distributions of two variates you may be able to ascertain what

* The term "variance" has recently been introduced by "Student" as a useful term for the squared standard deviation.

† *Biometrika*, Vol. XIV. pp. 355—377.

‡ Equations on Rhodes', p. 366 certainly suggest it.

their correlation ought to be! This must be my excuse for publishing Mr Rhodes' special case before those of Dr Filon, and that of Dr Isserlis.

I have, however, always felt that while departure from linearity of regression is by no means uncommon, and while it almost always involves heteroscedasticity, it is not unreasonable to assume in the first place simple functions for our regression curves, straight lines, parabolae, cubics, and similar curves for our scedastic functions. I am not, however, convinced that the scedastic curve should be a plot of standard deviation of array to variate determining the array, I believe it quite possible that simpler results will be obtained by plotting the variance, i.e. the second moment coefficient μ_2 for the array. For some years past it has been gradually growing in my mind, that the best way under existing circumstances of approaching the skew correlation problem is to assume the forms of the regression and the scedastic curves and deduce from these upon some reasonable hypothesis the equation to the frequency surface itself. I had realised from the functional equation that, with certain limitations, not only the scedasticity, but the form of the variate frequency surfaces depends upon the regression lines. Chance had thrown in my way Professor Seimatsu Narumi—a brilliant Japanese mathematician—and I put before him the problem of working from the functional equation with given regression and scedasticity functions, back to the frequency surface. His recent remarkable papers published in *Biometrika* demonstrate that this is to a mathematician of his powers a more or less opened door to the solution of the general types of skew frequency surfaces. I confess to not a little pleasure in the fact that two of his general types are surfaces we had already reached, but they are now given their lawful generality. The Filon-Isserlis surface is the only type with linear regression and linear heteroscedasticity; my surface is the only type with linear regression and parabolic variance. The Rhodes surface of course does not appear, for its regression lines are not forms from which one can naturally start. What is more the normal or Gaussian surface is the only surface which is at the same time homoscedastic and of linear regression both ways. These and many other of Narumi's results are based on the limitation that the array frequency distributions are similar curves, not similarly placed, but reduced in scale owing to the heteroscedasticity, or variation of the standard deviations of the arrays. That is to say when we have equalised the array populations and measured the array variate in terms of its individual standard deviation, the array frequencies will be represented by similar curves. In both the forms of surface to which we have just referred the array curves as well as the marginal totals are given by Pearson skew curves.

On the publication of Narumi's paper I had a letter from Dr Isserlis suggesting that his method of fitting a frequency surface of the type

$$z = z_0 (x + g_1)^{p_1} (y + g_2)^{p_2} \{(g_1 - a_2)(y + a_1) + (g_2 - a_1)(x + a_2)\}^q \quad \ldots\ldots\text{(i)},$$

using Narumi's notation, should now be published. If the Editor of this Journal admits the reasonableness of this, he is confronted by the still greater reasonableness of publishing Professor Filon's investigation of 1901. He is further faced by the fact that excellent as both these investigations are, and in several respects

supplementary, neither of them in his estimation gives as direct a process of fitting as that suggested by Professor Narumi's method of approaching the problem. Having regard to the heavy expense of printing anything at all now-a-days, it seems best to roll the investigations of our three mathematicians into one and give them an undivided credit for the whole. On the other hand and for reasons stated above I do not think the Filon-Isserlis surface will be of great practical service. Hence, as I am at present advised, I do not believe it needs treating at great length.

(2) Narumi takes as his regression lines

$$x = m_1 y + c_1,$$
$$y = m_2 x + c_2.$$

Hence with any origin the four constants m_1, m_2, c_1, c_2 are determined in the usual way by a knowledge of $\bar{x}$, $\bar{y}$, σ_x, σ_y and r_{xy}, these quantities having their usual significance.

Narumi takes for his scedastic lines

$$\sigma_{x_y} = \lambda_1 (y + a_1), \quad \sigma_{y_x} = \lambda_2 (x + a_2).$$

By calculating the standard deviations of the arrays and plotting them to y or x as the case may be, we determine whether the system has linear scedasticity, the best fitting lines to these points giving us a_1 and a_2. To obtain these scedastic straight lines and the regression lines is really needful *a priori* in order to satisfy ourselves of the fundamental assumptions of linearity in regression and scedasticity.

Narumi writes $g_1 = m_1 a_1 - c_1$ and $g_2 = m_2 a_2 - c_2$. Thus all the constants of equation (i) except the powers and z_0 are found by familiar processes, which are in themselves needful, not only to test our assumptions, but to exhibit our results.

By changing the origin* and writing $b_1 = 2(g_1 - a_2)$, $b_2 = 2(g_2 - a_1)$, and introducing a new constant z_0', we have

$$z = z_0' \left(\frac{x'}{b_1}\right)^{p_1} \left(\frac{y'}{b_2}\right)^{p_2} \left(1 - \frac{x'}{b_1} - \frac{y'}{b_2}\right)^{q},$$

which is Filon's standard form. Here we may drop the dashes and note that the mode x_m, y_m is given by

$$x_m = \frac{p_1 b_1}{q + p_1 + p_2}, \quad y_m = \frac{p_2 b_2}{q + p_1 + p_2}.$$

We have no trouble about the mean; it has already been determined and is the intersection of the two regression lines.

Clearly the whole surface lies in the area between the three lines $x = 0$, $y = 0$, $\frac{x}{b_1} + \frac{y}{b_2} - 1 = 0$, and according to the signs of p_1, p_2 and q will be zero at these lines or will asymptote to vertical planes drawn through them. Filon indicates by a graph the contour lines, which are very similar to the polygon-contours in whist†.

* I.e. write $x + g_1 = x'$, $y + g_2 = y'$.

† See *Phil. Trans.* Vol. 186 A, Plate 9.

The process is now very easy:

$$z_x = z_0 \int_0^{b_2\left(1-\frac{x}{b_1}\right)} \left(\frac{y}{b_2}\right)^{p_2} \left(1 - \frac{x}{b_1} - \frac{y}{b_2}\right)^q dy \times \left(\frac{x}{b_1}\right)^{p_1}.$$

Write $$y'' = \frac{y}{b_2} \Big/ \left(1 - \frac{x}{b_1}\right)$$

and we have

$$z_x = z_0 b_2 \left(\frac{x}{b_1}\right)^{p_1} \left(1 - \frac{x}{b_1}\right)^{p_2+q+1} \int_0^1 y''^{p_2}(1-y'')^q \, dy''$$

$$= z_0 b_2 B(p_2+1, q+1) \left(\frac{x}{b_1}\right)^{p_1} \left(1 - \frac{x}{b_1}\right)^{p_2+q+1} \quad \text{.....(ii).}$$

This equation shows that the marginal totals as well as the arrays are Pearson curves of Type I, or its special derivatives. Proceeding to another integration we have, if N be the total population,

$$N = z_0 b_1 b_2 B(p_2+1, q+1) B(p_1+1, p_2+q+2)$$

$$= z_0 b_1 b_2 \frac{\Gamma(p_1+1)\Gamma(p_2+1)\Gamma(q+1)}{\Gamma(p_1+p_2+q+3)},$$

which gives

$$z_0 = \frac{N}{b_1 b_2} \frac{\Gamma(p_1+p_2+q+3)}{\Gamma(p_1+1)\Gamma(p_2+1)\Gamma(q+1)} \quad \text{.....(iii).}$$

We now only require the constants p_1, p_2 and q. These are to be found in the ordinary way by fitting (ii), i.e. curves of Type I, to the marginal totals*.

We know† that if we have any curve of Type I, i.e.

$$y = y_0 \left(\frac{x}{b}\right)^{m_1} \left(1 - \frac{x}{b}\right)^{m_2} \quad \text{.....(iv),}$$

where b is the range, its mean is given by

$$\mu_1' = \frac{b(m_1+1)}{m_1+m_2+2}$$

from the start as origin.

Its standard deviation σ is determined by

$$\sigma^2 = \frac{b^2(m_1+1)(m_2+1)}{(m_1+m_2+2)^2(m_1+m_2+3)} \quad \text{.....(v),}$$

and its mode is given by

$$x_m = \frac{m_1}{m_1+m_2} b \quad \text{.....(vi).}$$

These results will enable us to study the chief characteristics of our surface.

When y is constant, the equation to a y-array of x's is

$$z_y = \text{constant} \times \left(\frac{x}{b_1}\right)^{p_1} \left(1 - \frac{x}{b_1\left(1-\frac{y}{b_2}\right)}\right)^q \quad \text{.....(vii).}$$

* Filon uses our $\sqrt{\beta_1}$ and $\sqrt{\beta_1'}$, and Isserlis β_1, β_1' to determine p_1, p_2 and q. The analysis seems longer, but appears to save the use of β_2, β_2'. However, these must be computed to determine whether the condition in (x) p. 227 is reasonably satisfied.

† *Phil. Trans. loc. cit.* pp. 367—370.

Accordingly: the range of the y-array of x's is

$$b_1\left(1-\frac{y}{b_2}\right).$$

The line of means is

$$\tilde{x}_y = b_1\left(1-\frac{y}{b_2}\right)\frac{p_1+1}{p_1+q+2} \quad \text{.....(viii),}$$

where $\tilde{x}_y$ is measured from $x=0$; the regression is thus linear as we settled *a priori*.

Accordingly
$$-\frac{b_1}{b_2}\frac{p_1+1}{p_1+q+2} = r_{xy}\frac{\sigma_x}{\sigma_y},$$

but similarly
$$-\frac{b_2}{b_1}\frac{p_2+1}{p_2+q+2} = r_{xy}\frac{\sigma_y}{\sigma_x}.$$

Hence by multiplying

$$r^2_{xy} = \frac{(p_1+1)(p_2+1)}{(p_1+q+2)(p_2+q+2)},$$

in other words:
$$r_{xy} = \pm\sqrt{\frac{(p_1+1)(p_2+1)}{(p_1+q+2)(p_2+q+2)}} \quad \text{.....(ix).}$$

The sign will be determined by the sign of the product moment, $S(x-\bar{x})(y-\bar{y})$. We thus see that r_{xy} the correlation depends solely on the constants of the marginal totals. But if the curve be given by (iv), we know that

$$m_1+m_2+2 = \frac{6(\beta_2-\beta_1-1)}{3\beta_1-2\beta_2+6},$$

or applying this to our curve of marginal totals

$$\frac{6(\beta_2-\beta_1-1)}{3\beta_1-2\beta_2+6} = p_1+p_2+q+3 = \frac{6(\beta_2'-\beta_1'-1)}{3\beta_1'-2\beta_2'+6},$$

where the undashed β's are for the x- and the dashed β's for the y-marginal totals. This result reduces to the simple form

$$\frac{\beta_2+3}{\beta_1+4} = \frac{\beta_2'+3}{\beta_1'+4} \quad \text{.....(x).}$$

Now we can determine from the β_1 and β_2 of the marginal total both p_1 and p_1+q; in fact if*

$$\chi_1 = \frac{6(\beta_2-\beta_1-1)}{3\beta_1-2\beta_2+6}, \quad \chi_2 = \frac{\chi_1^2}{4+\frac{1}{4}\beta_1(\chi_1+2)^2/(\chi_1+1)},$$

p_1+1 and p_2+q+2 are roots of the quadratic

$$\lambda^2-\chi_1\lambda+\chi_2=0.$$

Similarly we have a quadratic

$$\lambda'^2-\chi_1\lambda+\chi_2=0,$$

where
$$\chi_2 = \frac{\chi_1^2}{4+\frac{1}{4}\beta_1'(\chi_1+2)^2/(\chi_1+1)},$$

to determine p_2+1 and p_1+q+2.

* *loc. cit.* pp. 368—9.

Thus, as I have previously stated, r_{xy} is a function solely of the variation constants, the β's of the distributions of the two variates, and if we knew the variates followed this type of correlation surface we could determine the correlation of two variates from their individual frequency distributions. This binding of correlation to variation seemed to me in 1901 and still seems a most improbable result. For example, if barometric frequencies followed this result*, we could ascertain from the pressure distributions at two stations what was the correlation between their pressures. I am open to conviction that correlation is a function of variation, but I do not think the relation probable in any large number of cases, and so the Filon-Isserlis surface does not appear to me to be likely to be of any great physical importance.

We may consider one or two further points with regard to it.

Turning back to equations (vi) and (vii) we see that the modes of arrays of x for given y are given by

$$x_{m,y} = \frac{p_1}{p_1+q} b_1 \left(1 - \frac{y}{b_2}\right) \quad \text{.....(xi),}$$

or, the curve of modes is linear.

Again the variance of an array of x's is:

$$\sigma^2_{x_y} = b_1^2 \left(1 - \frac{y}{b_2}\right)^2 \frac{(p_1+1)(q+1)}{(p_1+q+2)^2(p_1+q+3)},$$

or,
$$\sigma_{x_y} = b_1 \left(1 - \frac{y}{b_2}\right) \frac{\sqrt{(p_1+1)(q+1)}}{(p_1+q+2)\sqrt{p_1+q+3}} \quad \text{.....(xii).}$$

This by aid of (viii) gives us:

$$\sigma_{x_y} = \bar{x}_y \frac{\sqrt{q+1}}{\sqrt{p_1+1}\sqrt{p_1+q+3}} \quad \text{.....(xiii).}$$

Thus the scedastic curve is linear and the array standard deviation varies directly as its mean.

A characteristic of the surface is accordingly (since $\bar{x}_y$ is measured from zero) that the coefficients of variation of the arrays (i.e. $100\sigma_{x_y}/\bar{x}_y$) are constant.

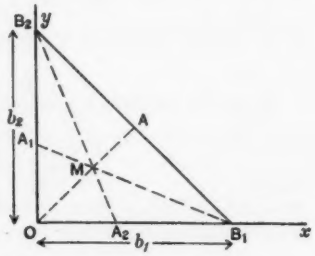
The mode of the surface as a whole is given by:

$$\tilde{x} = b_1 \frac{p_1}{p_1+p_2+q}, \quad \tilde{y} = b_2 \frac{p_2}{p_1+p_2+q} \quad \text{.....(xiv).}$$

If the fundamental triangle be OB_1B_2 and M the mode be joined to OB_1B_2, then the lines OMA, B_1MA_1 and B_2MA_2 are the lines of modes of the sections parallel respectively to B_1B_2, OB_2 and OB_1. These lines are accordingly the locus of the points of contact of tangents parallel respectively to B_1B_2, i.e.

$$\frac{y}{b_2} + \frac{x}{b_1} - 1 = 0,$$

Oy and Ox. As soon as this is realised the nature of the trianguloid ovals which serve for contours becomes fairly clear.



* They certainly do not as the quantity $(\beta_2+3)/(\beta_1+4)$ is not constant for all stations.

Filon extended the theory in 1901 to multiple correlation of the same type. He took for his equation to the multiple surface:

$$z_{12\ldots n} = z_0 \left(\frac{x_1}{b_1}\right)^{p_1} \left(\frac{x_2}{b_2}\right)^{p_2} \ldots \left(\frac{x_n}{b_n}\right)^{p_n} \left(1 - \frac{x_1}{b_1} - \frac{x_2}{b_2} - \ldots - \frac{x_n}{b_n}\right)^{t_n}.$$

By successive integrations he reduced this to

$$z_{12} = z_0 b_3 b_4 \ldots b_n \left(\frac{x_1}{b_1}\right)^{p_1} \left(\frac{x_2}{b_2}\right)^{p_2} \left(1 - \frac{x_1}{b_1} - \frac{x_2}{b_2}\right)^{p_3+p_4+\ldots+p_n+t_n+n-2} \times \frac{\Gamma(p_3+1)\,\Gamma(p_4+1)\ldots\Gamma(p_n+1)\,\Gamma(t+1)}{\Gamma(p_3+p_4+\ldots+p_n+t_n+n-1)}.$$

Hence using (ix) we find:

$$r_{12} = \frac{\sqrt{(p_1+1)(p_2+1)}}{\sqrt{p_1+p_3+p_4+\ldots+p_n+t_n+n}\,\sqrt{p_2+p_3+p_4+\ldots+p_n+t_n+n}},$$

or, writing

$$s_n = p_1 + p_2 + \ldots + p_n + t_n + n,$$

$$r_{12} = \sqrt{\frac{(p_1+1)(p_2+1)}{(s_n-p_1)(s_n-p_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xv)}.$$

Now the correlation of two variates ought to be independent of n, i.e. (xv) should give the same result as (ix). It is needful therefore that s_n should be a constant or we must take

$$t_n = m - n - p_1 - p_2 - \ldots - p_n,$$

which gives $s_n =$ some constant, m.

Thus the general correlation surface for n variates of the present type is

$$z = z_0 \left(\frac{x_1}{b_1}\right)^{p_1} \left(\frac{x_2}{b_2}\right)^{p_2} \ldots \left(\frac{x_n}{b_n}\right)^{p_n} \left(1 - \frac{x_1}{b_1} - \frac{x_2}{b_2} - \frac{x_3}{b_3} - \ldots - \frac{x_n}{b_n}\right)^{m-n-\Sigma(p)} \quad \ldots\text{(xvi)}.$$

The limitations of this system of multiple correlation are very great however. We have

$$r_{uv} = \sqrt{\frac{(p_u+1)(p_v+1)}{(m-p_u)(m-p_v)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xvii)}.$$

Every correlation is accordingly dependent upon the β_1, β_2 of the two variates and the condition of (x) must hold for each pair

$$\frac{\beta_2+3}{\beta_1+4} = \frac{\beta_2'+3}{\beta_1'+4} = \frac{\beta_2''+3}{\beta_1''+4} = \ldots = \frac{\beta_2^{(n)}+3}{\beta_1^{(n)}+4}.$$

Further each correlation involves m and two p's. Thus if there are n variates or $\frac{1}{2}n(n-1)$ correlation coefficients, these coefficients are functions of $n+1$ constants m and the n p's. But these $n+1$ constants do not appear in an independent form; for, if we write $p_u' = p_u + 1$, $p_v' = p_v + 1$, $m' = m + 1$, we have

$$r_{uv} = \sqrt{\frac{1}{\frac{m'}{p_u'} - 1}\,\frac{1}{\frac{m'}{p_v'} - 1}},$$

or if $1\Big/\left(\frac{m'}{p_u'}-1\right)=\lambda_u$ be calculated for each marginal total from its β's we shall have

$$r_{uv}=\sqrt{\lambda_u\lambda_v},$$

or, there really are only n independent constants for our $\frac{1}{2}n(n-1)$ correlation coefficients, or there will always be $\frac{1}{2}n(n-3)$ relations between the correlation coefficients. For example, if we have four variates, we must have two conditions and these are easily found to be:

$$r_{13}r_{24}=r_{14}r_{23}=r_{12}r_{34}.$$

If we have five variates, we have five conditions, which may be expressed in several ways but are involved in:

$$r_{12}r_{34}=r_{14}r_{23}=r_{13}r_{24}=r_{42}r_{13},$$
$$r_{12}r_{35}=r_{32}r_{15},$$
$$r_{13}r_{45}=r_{42}r_{15}.$$

It may be possible that for certain correlation systems—especially those based on games of chance—such relations may hold, or even in various theories of central mental factors. But it is hard to believe that it is advisable to start fitting a general frequency distribution with a surface subject to such arbitrary limitations.

ON NON-SKEW FREQUENCY SURFACES.

By KARL PEARSON, F.R.S.

(1) The surfaces discussed in this paper were briefly referred to at a meeting of the *Society of Biometricians and Mathematical Statisticians* on May 15, 1922 when Mr E. C. Rhodes' paper was read*. In 1913† I had pointed out that with linear regression the two ways of determining $p_{x^2y^2}$ suggested that for a whole range of bi-variate frequency surfaces β_2 (for x) and β_2' (for y) would be equal, and that there existed in such cases a long series of relations among the still higher β's. I then set about investigating the range of frequency surfaces which would have symmetrical marginal totals, i.e. cases in which β_1 and β_1' would be zero for both marginal totals. In my system of frequency curves the marginal totals would accordingly have to follow distributions of the form

$$z = z_0 \left(1 \pm \frac{x^2}{b^2}\right)^{\pm n} \quad \text{.....(i),}$$

which includes the normal curve as a special case and has some interesting particular cases, which have been discussed elsewhere‡. I started my investigations by assuming the required surfaces to be of the form

$$z = z_0 \Big/ \left(1 + b_1 \frac{x^2}{\sigma_1^2} + 2b_2 \frac{xy}{\sigma_1\sigma_2} + b_3 \frac{y^2}{\sigma_2^2} + b_4 \frac{x^2y^2}{\sigma_1^2\sigma_2^2}\right)^n \quad \text{.....(ii),}$$

where x and y are the two variates, σ_1, σ_2 their standard deviations, the origin is at the means, and $z\delta x\delta y$ represents as usual the frequency of individuals with characters between x and $x+\delta x$, y and $y+\delta y$. Clearly both x-arrays of y's (x = constant) and y-arrays of x's (y = constant) will be curves of the form (i). but under what conditions will the marginal totals also be of this form? Now before we can discuss (ii) we must make certain assumptions: we might suppose that the limits l_1, l_1' of a y-array of x's are independent of y and l_2, l_2' of an x-array of y's are independent of x. Let $\phi_1(x)$ be the function giving the x-marginal totals and $\phi_2(y)$ that giving the y-marginal totals. Then we have

$$\phi_1(x) = \int_{l_2}^{l_2'} z\,dy = z_0 \int_{l_1}^{l_2} \frac{dy}{\left(1 + b_1 \frac{x^2}{\sigma_1^2} + 2b_2 \frac{xy}{\sigma_1\sigma_2} + b_3 \frac{y^2}{\sigma_2^2} + b_4 \frac{x^2y^2}{\sigma_1^2\sigma_2^2}\right)^n}.$$

Now let us write

$$Y = \frac{y}{\sigma_2} + b_2 \frac{x}{\sigma_1} \frac{1}{b_3 + b_4 \frac{x^2}{\sigma_1^2}},$$

* See *Biometrika*, Vol. XIV. pp. 355—77.

† See *Biometrika*, Vol. IX. p. 4.

‡ *Phil. Trans.* Vol. 216A, pp. 448—50.

then
$$\phi_1(x) = z_0\sigma_2 \int_{L_2}^{L_2'} \frac{dY}{\left(1 + b_1\frac{x^2}{\sigma_1^2} - \frac{b_2^2\frac{x^2}{\sigma_1^2}}{b_3 + b_4\frac{x^2}{\sigma_1^2}} + \left(b_3 + b_4\frac{x^2}{\sigma_1^2}\right)Y^2\right)^n} \quad \text{.....(iii),}$$

where
$$\left.\begin{matrix} L_2 \\ L_2' \end{matrix}\right\} = \frac{1}{\sigma_2}\left\{\begin{matrix} l_2 \\ l_2' \end{matrix}\right\} + \frac{1}{\sigma_1}\frac{b_2 x}{b_3 + b_4\frac{x^2}{\sigma_1^2}}.$$

It is clear that L_2 and L_2' will no longer be generally independent of x. Now let us take

$$Y' = \frac{\sqrt{b_3 + b_4\frac{x^2}{\sigma_1^2}}}{\left(1 + b_1\frac{x^2}{\sigma_1^2} - \frac{b_2^2\frac{x^2}{\sigma_1^2}}{b_3 + b_4\frac{x^2}{\sigma_1^2}}\right)^{\frac{1}{2}}} Y = c_x Y.$$

Then we have
$$\phi_1(x) = z_0\sigma_2 \frac{\left(b_3 + b_4\frac{x^2}{\sigma_1^2}\right)^{n-1}\int_{\lambda_2}^{\lambda_2'}\frac{dY'}{(1+Y'^2)^n}}{\left(b_3 + (b_1b_3 + b_4 - b_2^2)\frac{x^2}{\sigma_1^2} + b_1b_4\frac{x^4}{\sigma_1^4}\right)^{n-\frac{1}{2}}} \quad \text{.....(iv),}$$

where
$$\left.\begin{matrix} \lambda_2 \\ \lambda_2' \end{matrix}\right\} = \frac{b_3 + b_4\frac{x^2}{\sigma_1^2}}{\left(b_3 + (b_1b_3 + b_4 - b_2^2)\frac{x^2}{\sigma_1^2} + b_1b_4\frac{x^4}{\sigma_1^4}\right)^{\frac{1}{2}}}\left[\frac{1}{\sigma_2}\left\{\begin{matrix} l_2 \\ l_2' \end{matrix}\right\} + \frac{1}{\sigma_1}\frac{b_2 x}{b_3 + b_4\frac{x^2}{\sigma_1^2}}\right] \text{...(v).}$$

Now if λ_2 and λ_2' are functions of x, the integral

$$\int_{\lambda_2}^{\lambda_2'}\frac{dY'}{(1+Y'^2)^n}$$

will be a function of x and it is exceedingly unlikely that $\phi_1(x)$ will take the form (i) owing to the presence in λ_2, λ_2' of the term b_2x. One solution would be to make l_2 and l_2' plus and minus infinity respectively, in which case the integral

$$\int_{\lambda_2}^{\lambda_2'}\frac{dY'}{(1+Y'^2)^n} = 2\int_0^{\infty}\frac{dY'}{(1+Y'^2)^n} = 2\int_0^{\frac{\pi}{2}}\cos^{2n-2}\theta\,d\theta = 2I_{2n-2},$$

and has been tabled*.

We have then

$$\phi_1(x) = 2z_0\sigma_2 I_{2n-2}\frac{\left(b_3 + b_4\frac{x^2}{\sigma_1^2}\right)^{n-1}}{\left(b_3 + (b_1b_3 + b_4 - b_2^2)\frac{x^2}{\sigma_1^2} + b_1b_4\frac{x^4}{\sigma_1^4}\right)^{n-\frac{1}{2}}} \quad \text{.....(vi).}$$

In order that this may reduce to the form (i) it is needful that either $b_4 = 0$, or that

$$\frac{b_2^2}{b_3} = \frac{2b_2b_4}{b_1b_3 + b_4 - b_2^2} = \frac{b_4^2}{b_1b_4}.$$

* See *Biometrika*, Vol. xi. p. 377.

These involve $b_4 = b_1 b_3$ and $b_2 = 0$. But under these circumstances equation (ii) becomes

$$z = z_0 \frac{1}{\left(1 + b_1 \frac{x^2}{\sigma_1^2}\right)^n} \frac{1}{\left(1 + b_3 \frac{y^2}{\sigma_2^2}\right)^n} \quad \text{.....(vii)}$$

or the two variates are independent and there is no correlation. This case has no interest for us. For l_2 and l_2' infinite we are left with the case $b_4 = 0$.

If we wish to give finite values to l_2 and l_1 and yet make λ_2 and λ_2' constant, so as to free ourselves from x in the integral term of (iv) we must make the factor in the square brackets on the right of (v) constant. This again leads to $b_4 = 0$, or $b_2 = 0$. The latter is as before unacceptable. It seems probable accordingly that nothing will be gained for our purposes by introducing the term b_4. For although it would allow of all arrays being curves for which β_1 would be zero, it would give a much more complicated form than (i) for the marginal totals.

(2) Let us therefore assume $b_4 = 0$ and the limits of the arrays to be $\pm \infty$.

In this case $$z = z_0 \Big/ \left(1 + b_1 \frac{x^2}{\sigma_1^2} + 2b_2 \frac{xy}{\sigma_1 \sigma_2} + b_3 \frac{y^2}{\sigma_2^2}\right)^n \quad \text{.....(viii)}$$

and the marginal frequencies are given by

$$\phi_1(x) = 2z_0 \sigma_2 I_{2n-2} \frac{b_3^{n-1}}{\left(b_3 + (b_1 b_3 - b_2^2)\frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}} \quad \text{.....(ix)}$$

$$= \frac{2z_0 \sigma_2 I_{2n-2}}{\sqrt{b_3}} \frac{1}{\left(1 + \frac{b_1 b_3 - b_2^2}{b_3} \frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}} \quad \text{.....(ix) } bis.$$

If N be the total frequency, we have

$$N = \int_{-\infty}^{+\infty} \phi_1(x)\, dx = \frac{2z_0 \sigma_2 I_{2n-2}}{\sqrt{b_3}} \int_{-\infty}^{+\infty} \frac{dx}{\left(1 + \frac{b_1 b_3 - b_2^2}{b_3} \frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}}$$

$$= \frac{4z_0 \sigma_1 \sigma_2}{\sqrt{b_1 b_3 - b_2^2}} I_{2n-2} I_{2n-3}.$$

But it is easy to prove that

$$I_{2n-2} I_{2n-3} = \frac{1}{2n-2} \frac{\pi}{2} \quad \text{.....(x)}$$

and accordingly $$z_0 = \frac{N}{\pi} \frac{(n-1)\sqrt{b_1 b_3 - b_2^2}}{\sigma_1 \sigma_2} \quad \text{.....(xi).}$$

Clearly the curves of equal frequency are a system of concentric, similar and similar placed ellipses, and since the frequency may be written

$$z = z_0 \Big/ \left\{1 + \left(b_1 \frac{x}{\sigma_1} + \frac{b_2}{b_1} \frac{y}{\sigma_2}\right)^2 + \frac{y^2}{\sigma_2^2}\left(b_3 - \frac{b_2^2}{b_1}\right)\right\}^n,$$

it follows that when y is constant, the array curve is symmetrical about the point

$$x = -\sigma_1 \frac{b_2}{b_1}\frac{y}{\sigma_2}.$$

This point must therefore be its mean, or the line of means is straight and given by

$$\bar{x}_y = -\frac{b_2}{b_1}\frac{\sigma_1}{\sigma_2}y \quad \text{.....(xii).}$$

Similarly $$\bar{y}_x = -\frac{b_2}{b_3}\frac{\sigma_2}{\sigma_1}x \quad \text{.....(xii) } bis.$$

Thus the regression both ways is linear and we must have, if r be the coefficient of correlation,

$$r = -\frac{b_2}{b_1} = -\frac{b_2}{b_3} \quad \text{.....(xiii),}$$

and thus $$b_1 = b_3.$$

Returning to equation (ix) *bis* we have

$$N\sigma_1^2 = \int_{-\infty}^{+\infty} \phi_1(x)\, x^2\, dx$$

$$= \frac{4z_0\sigma_2 I_{2n-2}}{\sqrt{b_3}} \int_0^\infty \frac{x^2\,dx}{\left(1 + \dfrac{b_1b_3 - b_2^2}{b_3}\dfrac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}}$$

$$= \frac{4z_0\sigma_2\sigma_1^3 I_{2n-2} b_3}{(b_1b_3 - b_2^2)^{\frac{3}{2}}} \int_0^\infty \frac{x'^2\,dx'}{(1+x'^2)^{n-\frac{1}{2}}}.$$

Or $$N = \frac{4z_0\sigma_2\sigma_1 I_{2n-2} b_3}{(b_1b_3 - b_2^2)^{\frac{3}{2}}} \left\{ \int_0^\infty \frac{dx'}{(1+x'^2)^{n-\frac{3}{2}}} - \int_0^\infty \frac{dx'}{(1+x'^2)^{n-\frac{1}{2}}} \right\}$$

$$= \frac{4z_0\sigma_2\sigma_1 I_{2n-2} b_3}{(b_1b_3 - b_2^2)^{\frac{3}{2}}} (I_{2n-3} - I_{2n-3}).$$

But $$I_{2n-3} = \frac{2n-3}{2n-4} I_{2n-3} \quad \text{.....(xiv).}$$

Whence using (x) and (xi) we deduce

$$1 = \frac{b_3}{b_1b_3 - b_2^2}\frac{1}{2(n-2)} \quad \text{..... .....(xv).}$$

Thus $$b_1 = \frac{b_1b_3}{b_1b_3 - b_2^2}\frac{1}{2(n-2)} = \frac{1}{1-r^2}\frac{1}{2(n-2)} \text{ by (xiii).}$$

Accordingly $$b_2 = -rb_1 = -\frac{r}{1-r^2}\frac{1}{2(n-2)}$$

and $$b_3 = \frac{1}{1-r^2}\frac{1}{2(n-2)} = b_1.$$

Thus the equation to our surface is

$$z = \frac{N}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-r^2}}\frac{n-1}{n-2} \frac{1}{\left\{1 + \dfrac{1}{2(n-2)}\dfrac{1}{1-r^2}\left(\dfrac{x^2}{\sigma_1^2} - \dfrac{2rxy}{\sigma_1\sigma_2} + \dfrac{y^2}{\sigma_2^2}\right)\right\}^n}$$

.....(xvi).

When n is infinite, this clearly becomes the ordinary normal surface for two variates.

We have now to consider the determination of n. Let us turn back to (ix) *bis*; we may write it in the form

$$\phi_1(x) = \frac{\sqrt{2}N}{\pi\sigma_1}\frac{n-1}{\sqrt{n-2}} I_{2n-2} \frac{1}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}} \quad \text{.....(xvii).}$$

Then
$$\mu_4 = \sigma_1^4\beta_2 = \frac{2\sqrt{2}}{\pi\sigma_1}\frac{n-1}{\sqrt{n-2}} I_{2n-2}\int_0^\infty \frac{x^4dx}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}}.$$

Or
$$\beta_2 = \frac{16(n-2)^2}{\pi} I_{2n-2}\int_0^\infty \frac{x'^4dx'}{(1+x'^2)^{n-\frac{1}{2}}}$$

$$= \frac{16(n-2)^2(n-1)}{\pi} I_{2n-2}\int_0^\infty \left\{\frac{1}{(1+x'^2)^{n-\frac{5}{2}}} - \frac{2}{(1+x'^2)^{n-\frac{3}{2}}} + \frac{1}{(1+x'^2)^{n-\frac{1}{2}}}\right\} dx$$

$$= \frac{16(n-2)^2(n-1)}{\pi} I_{2n-2}(I_{2n-7} - 2I_{2n-5} + I_{2n-3})$$

$$= \frac{16(n-2)^2(n-1)}{\pi} I_{2n-2}I_{2n-3}\left(\frac{(2n-5)(2n-3)}{(2n-6)(2n-4)} - 2\frac{2n-3}{2n-4} + 1\right)$$

$$= \frac{16(n-2)^2}{\pi}\frac{n-1}{2n-2}\frac{\pi}{2}\frac{3}{4(n-3)(n-2)} = \frac{3(n-2)}{n-3}.$$

Accordingly
$$n = \frac{3(\beta_2-2)}{\beta_2-3} \quad \text{.....(xviii).}$$

Similarly
$$n = \frac{3(\beta_2'-2)}{\beta_2'-3},$$

and it follows that $\beta_2' = \beta_2$, or both variates must have the same β_2. This is the somewhat serious limitation of this non-skew regression surface; it will only be applicable to symmetrical variates of homologous character, e.g. the same character in adult relatives.

Finally we have

$$z = \frac{N}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-r^2}}\frac{2\beta_2-3}{\beta_2}\frac{1}{\left\{1+\frac{\beta_2-3}{2\beta_2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)\right\}^{\frac{3(\beta_2-2)}{\beta_2-3}}} \quad \text{.....(xix),}$$

$$\left.\begin{aligned}\phi_1(x) &= \frac{\sqrt{2}N}{\pi\sigma_1}\frac{2\beta_2-3}{\sqrt{\beta_2(\beta_2-3)}} I_{2n-2}\frac{1}{\left(1+\frac{\beta_2-3}{2\beta_2}\frac{x^2}{\sigma_1^2}\right)^{\frac{5\beta_2-9}{\beta_2-3}}}\\ \phi_2(y) &= \frac{\sqrt{2}N}{\pi\sigma_2}\frac{2\beta_2-3}{\sqrt{\beta_2(\beta_2-3)}} I_{2n-2}\frac{1}{\left(1+\frac{\beta_2-3}{2\beta_2}\frac{y^2}{\sigma_2^2}\right)^{\frac{5\beta_2-9}{\beta_2-3}}}\end{aligned}\right\} \quad \text{.....(xx),}$$

which give the equations to the surface and to the marginal totals. These hold as long as $\beta_2 > 3$, and when $\beta_2 = 3$, the surface breaks down into the normal frequency surface, which is most easily seen from (vi). For (xx) in general the value of I_{2n-2} may be found with fair accuracy by interpolation from the table in *Biometrika*, Vol. XI. p. 377. Otherwise we have

$$\begin{aligned} I_{2n-2} &= \int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(1-\sin^2\theta)^{\frac{2n-3}{2}}}{\sqrt{\sin^2\theta}}\, d\sin^2\theta \\ &= \frac{1}{2}\int_0^1 (1-x)^{n-\frac{3}{2}} x^{-\frac{1}{2}}\, dx = \tfrac{1}{2} B\left(n-\tfrac{1}{2}, \tfrac{1}{2}\right) \\ &= \frac{1}{2}\frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n)} \\ &= \frac{\sqrt{\pi}}{2}\frac{1}{n-1}\frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n-1)}. \end{aligned}$$

But an application of Stirling's Theorem shows us that $\Gamma(n-\frac{1}{2})/\Gamma(n-1)$ approaches the limit $\sqrt{n}$ as n is increased. Accordingly

$$\frac{\sqrt{2}}{\pi}\frac{n-1}{\sqrt{n-2}} I_{2n-2} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{n-2}},$$

or, $\phi_1(x)$ approaches

$$\frac{N}{\sqrt{2\pi}\sigma_1}\sqrt{\frac{n}{n-2}}\frac{1}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}} = \frac{N}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\frac{x^2}{\sigma_1^2}},$$

as n increases indefinitely. More generally

$$\phi_1(x) = \frac{N}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{n-2}}\frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n-1)}\frac{1}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}} \ldots\ldots (\text{xx})\, bis,$$

which permits of evaluation from tables of the complete Γ-function*, as soon as n is known from (xviii).

We have seen that both regressions are linear. We now ask what is the nature of the scedasticity of the arrays.

We may write the surface in the form:

$$z = \frac{N}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\frac{n-1}{n-2}\frac{1}{\left\{1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}+\frac{1}{2(n-2)}\frac{1}{\sigma_2^2(1-r^2)}\left(y-\frac{\sigma_2 r x}{\sigma_1}\right)^2\right\}^n}.$$

Let
$$Y = y - \frac{\sigma_2 r x}{\sigma_1},$$

then for the standard deviation of an array σ_{y_x} we have

$$\phi_1(x)\,\sigma^2_{y_x} = \frac{2N}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\frac{n-1}{n-2}\int_0^\infty \frac{Y^2 dY}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}+\frac{1}{2(n-2)(1-r^2)}\frac{Y^2}{\sigma_2^2}\right)^n}.$$

* *Tracts for Computers*, Nos. VIII and IX, Cambridge University Press.

Now let $$\frac{1}{\{2(n-2)\}^{\frac{1}{2}}}\frac{Y}{\sigma_2\sqrt{1-r^2}}=\sqrt{1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}}\,Y',$$

and we find

$$\phi_1(x)\,\sigma^2_{y_x}=\frac{N}{\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-r^2}}\frac{n-1}{n-2}\frac{\sigma_2^2\{2(n-2)\}^{\frac{1}{2}}(1-r^2)^{\frac{3}{2}}}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{3}{2}}}\int_0^\infty\frac{Y'^2dY'}{(1+Y'^2)^n}.$$

Thus
$$\begin{aligned}\sigma^2_{y_x}&=\sigma_2^2\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)\frac{(1-r^2)\,2(n-2)}{I_{2n-2}}(I_{2n-4}-I_{2n-2})\\&=\sigma_2^2\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)(1-r^2)\,2(n-2)\left(\frac{2n-2}{2n-3}-1\right)\\&=\sigma_2^2(1-r^2)\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)\frac{2n-4}{2n-3}.\end{aligned}$$

Accordingly
$$\sigma^2_{y_x}=\sigma_2^2(1-r^2)\left\{1-\frac{1}{2n-3}\left(1-\frac{x^2}{\sigma_1^2}\right)\right\}\text{.....(xxi)},$$

and similarly
$$\sigma^2_{x_y}=\sigma_1^2(1-r^2)\left\{1-\frac{1}{2n-3}\left(1-\frac{y^2}{\sigma_2^2}\right)\right\}\text{ .....(xxi) \textit{bis}.}$$

Now the mean square deviations from the regression lines should be $\sigma_2^2(1-r^2)$ and $\sigma_1^2(1-r^2)$ respectively, or it follows that

$$\int_0^\infty\frac{\phi_1(x)}{N}\left(1-\frac{x^2}{\sigma_1^2}\right)dx$$

must be zero.

But
$$1-\frac{x^2}{\sigma_1^2}=2(n-2)\left\{\frac{2n-3}{2(n-2)}-\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)\right\}.$$

Substituting for $\phi_1(x)$ we find that

$$\frac{2n-3}{2(n-2)}\int_0^\infty\frac{dx}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{1}{2}}}-\int_0^\infty\frac{dx}{\left(1+\frac{1}{2(n-2)}\frac{x^2}{\sigma_1^2}\right)^{n-\frac{3}{2}}}$$

must be zero, or

$$\frac{2n-3}{2(n-2)}I_{2n-3}\text{ must equal }I_{2n-3},$$

which is an obvious truth since

$$I_{m-2}=\frac{m}{m-1}I_m.$$

The advantage of this manner of looking at the result is that we grasp that, while the scatter becomes infinite with x and y, still the frequency is so small that it dominates the scatter as we approach infinity. Accordingly we perceive that for $x<\sigma_1$ the scatter of an array has less than its mean value; for $x=\sigma_1$ it takes its mean value $\sigma_2\sqrt{1-r^2}$ and for $x>\sigma_1$, the scatter of the array is greater than its mean value. The scedastic curve giving σ_{y_x} in terms of x is a hyperbola, i.e.

$$\frac{\sigma^2_{y_x}}{\frac{2(n-2)}{2n-3}\sigma_2^2(1-r^2)}-\frac{x^2}{2(n-2)\sigma_1^2}=1\text{ .....(xxii)},$$

with its axes in the direction of the principal variates and of magnitudes respectively

$$2\sqrt{2(n-2)}\,\sigma_1 \text{ and } 2\sqrt{\frac{2(n-2)}{2n-3}}\,\sigma_2\sqrt{1-r^2},$$

the real axis being that of y, or of σ_{y_x}. Similar properties hold for the scatter of x.

We can only make the system homoscedastic if we make n infinite or $\beta_2 = 3$, that is if we reduce the system to the normal surface.

(3) Let us now consider what happens if β_2 be less than 3. In this case n will be negative and our infinite limits for x and y will be inappropriate. We start therefore with the assumption that

$$z = z_0\left\{1 + b_1\frac{x^2}{\sigma_1^2} + 2b_2\frac{xy}{\sigma_1\sigma_2} + b_3\frac{y^2}{\sigma_2^2}\right\}^n,$$

where n is positive.

In order that the frequency should be finite, it is needful that the contour lines of equal frequency should be ellipses, and that z should be zero along one of these ellipses, within the area of which all the frequency should lie. The surface must therefore take the form

$$z = z_0\left\{1 - \frac{1}{(1-r^2)\,c^2}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right\}^n \text{.....(xxiii),}$$

where c^2 is a constant.

The regression lines are clearly straight lines, r will be the correlation coefficient, σ_1 and σ_2 the standard deviations as before. Writing

$$Y = y - \frac{\sigma_2}{\sigma_1}rx,$$

we have

$$z = z_0\left\{1 - \frac{1}{c^2}\frac{x^2}{\sigma_1^2} - \frac{Y^2}{c^2\sigma_2^2}\frac{1}{1-r^2}\right\}^n \text{.....(xxiii) } bis.$$

Now clearly the contour of zero frequency is the ellipse

$$(1-r^2)\,c^2 = \frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \text{.....(xxiv),}$$

and if we wish to obtain $\phi_1(x)$ we must integrate y (or Y) from one side to the other of this ellipse, i.e.

$$\phi_1(x) = z_0\int_{-a}^{+a}\left(1 - \frac{1}{c^2}\frac{x^2}{\sigma_1^2} - \frac{Y^2}{c^2\sigma_2^2(1-r^2)}\right)^n dY,$$

where

$$a = c\sigma_2\sqrt{1-r^2}\sqrt{1 - \frac{1}{c^2}\frac{x^2}{\sigma_1^2}}.$$

Accordingly if we take

$$Y = \left(1 - \frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)^{\frac{1}{2}} c\sigma_2\sqrt{1-r^2}\,Y',$$

we have

$$\phi_1(x) = 2z_0\int_0^1 (1-Y'^2)^n\,dY'\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)^{n+\frac{1}{2}} c\sigma_2\sqrt{1-r^2}$$

$$= 2z_0 c\sigma_2\sqrt{1-r^2}\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)^{n+\frac{1}{2}}\int_0^{\frac{\pi}{2}}\cos^{2n+1}\theta\,d\theta$$

$$= 2z_0 c\sigma_2\sqrt{1-r^2}\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)^{n+\frac{1}{2}} I_{2n+1} \quad \text{.....(xxv).}$$

If we now integrate this for the whole possible range of values of x, i.e. from $-c\sigma_1$ to $+c\sigma_1$, we have for the total frequency

$$N = 4z_0 c\sigma_2\sqrt{1-r^2}\int_0^{c\sigma_1}\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)^{n+\frac{1}{2}} dx\, I_{2n+1}$$

$$= 4z_0 c^2\sigma_1\sigma_2\sqrt{1-r^2} I_{2n+2} I_{2n+1}$$

$$= 4z_0 c^2\sigma_1\sigma_2\sqrt{1-r^2}\,\frac{\pi}{4n+4}.$$

Thus we have $$z_0 = \frac{N}{\pi\sigma_1\sigma_2\sqrt{1-r^2}}\,\frac{n+1}{c^2} \quad \text{.....(xxvi).}$$

We now turn back to (xxv) and if we determine the integral

$$\sigma_1^2 = \int_{-c\sigma_1}^{+c\sigma_1}\frac{\phi_1(x)\,x^2\,dx}{N},$$

find it is $$\sigma_1^2 c^2\,\frac{1}{2(n+2)},$$

or $$c^2 = 2(n+2) \quad \text{.....(xxvii).}$$

On the other hand

$$\beta_2 = \int_{-c\sigma_1}^{+c\sigma_1}\frac{\phi_1(x)\,x^4\,dx}{N\sigma_1^4} = \frac{2\sigma_1}{N}\int_0^{c\sigma_1}\phi_1(x)\left(\frac{x}{\sigma_1}\right)^4 d\left(\frac{x}{\sigma_1}\right)$$

$$= \frac{4}{\pi}\frac{1}{c} I_{2n+1}(n+1)\int_0^{c\sigma_1}\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)^{n+\frac{1}{2}}\left(\frac{x}{\sigma_1}\right)^4 d\left(\frac{x}{\sigma_1}\right);$$

or, if $x' = \dfrac{x}{c\sigma_1}$,

$$= \frac{4}{\pi}c^4 I_{2n+1}(n+1)\int_0^1 (1-x'^2)^{n+\frac{1}{2}}\{1-(1-x'^2)\}^2\,dx'$$

$$= \frac{4}{\pi}c^4(n+1)\,I_{2n+1}(I_{2n+2} - 2I_{2n+4} + I_{2n+6})$$

$$= \frac{4}{\pi}c^4(n+1)\,\frac{\pi}{4(n+1)}\left(1-2\,\frac{2n+3}{2n+4}+\frac{2n+3}{2n+4}\,\frac{2n+5}{2n+6}\right)$$

$$= c^4\,\frac{3}{4(n+2)(n+3)} = \frac{3(n+2)}{n+3} \text{ by (xxvii).}$$

Accordingly

$$\left.\begin{aligned}\beta_2 &= \frac{3(n+2)}{n+3}\\ \text{and}\qquad n &= \frac{3(\beta_2-2)}{3-\beta_2}\end{aligned}\right\}\quad\text{.....(xxviii).}$$

Thus we have

$$z = \frac{N}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\,\frac{n+1}{n+2}\left\{1-\frac{1}{2(n+2)}\,\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)\right\}^n \text{...(xxix),}$$

where $n = 3(\beta_2-2)/(3-\beta_2)$.

It will be seen that this form is exactly analogous to what we obtain by writing $-n$ for n in (xvi), but the proof by which we reached (xvi) would no longer be valid, for we should not have infinite limits for x and y.

Further, the marginal frequencies are given by

$$\phi_1(x) = \frac{\sqrt{2}N}{\pi\sigma_1}\,\frac{n+1}{\sqrt{n+2}}\left(1-\frac{x^2}{2(n+2)\sigma_1^2}\right)^{n+\frac{1}{2}} I_{2n+1} \text{.....(xxx),}$$

$$\phi_2(y) = \frac{\sqrt{2}N}{\pi\sigma_2}\,\frac{n+1}{\sqrt{n+2}}\left(1-\frac{y^2}{2(n+2)\sigma_2^2}\right)^{n+\frac{1}{2}} I_{2n+1} \text{.....(xxx) } bis.$$

Since

$$I_{2n+1} = \frac{1}{2}\int_0^1 (1-x)^n x^{-\frac{1}{2}}\,dx$$

$$= \tfrac{1}{2}\sqrt{\pi}\,\Gamma(n+1)/\Gamma(n+\tfrac{3}{2}),$$

we can write for the marginal frequencies

$$\phi_1(x) = \frac{N}{\sqrt{2\pi}\sigma_1}\,\frac{1}{\sqrt{n+2}}\,\frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})}\left(1-\frac{x^2}{2(n+2)\sigma_1^2}\right)^{n+\frac{1}{2}} \text{.....(xxxi),}$$

and similarly

$$\phi_2(y) = \frac{N}{\sqrt{2\pi}\sigma_2}\,\frac{1}{\sqrt{n+2}}\,\frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})}\left(1-\frac{y^2}{2(n+2)\sigma_2^2}\right)^{n+\frac{1}{2}} \text{...(xxxi) } bis.$$

Here

$$n = \frac{3(\beta_2-2)}{3-\beta_2},\quad n+\tfrac{1}{2} = \frac{5\beta_2-9}{2(3-\beta_2)},\quad n+2 = \frac{\beta_2}{3-\beta_2},\quad n+\tfrac{3}{2} = \frac{3(\beta_2-1)}{2(3-\beta_2)}.$$

We now pass to a consideration of the scedasticity of the arrays. We have

$$\sigma^2_{y_x} = \frac{\int_{-a}^{+a} z\,Y^2 dY}{\phi_1(x)} = \frac{\int_{-a}^{+a}\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}-\frac{Y^2}{c^2\sigma_2^2(1-r^2)}\right)^n Y^2 dY}{2c\sigma_2\sqrt{1-r^2}\,I_{2n+1}\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)^{n+\frac{1}{2}}}$$

by (xxiii) *bis* and (xxv).

As before by the transformation

$$Y = c\sigma_2\sqrt{1-r^2}\sqrt{1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}}\,Y'$$

we find

$$\sigma^2_{y_x} = c^2\sigma_2^2(1-r^2)\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)\frac{\int_0^1(1-Y'^2)^n Y'^{\frac{1}{2}}dY'}{I_{2n+1}}$$

$$= c^2\sigma_2^2(1-r^2)\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right)\frac{I_{2n+1}-I_{2n+3}}{I_{2n+1}}$$

$$= \frac{2(n+2)}{2n+3}\sigma_2^2(1-r^2)\left(1-\frac{1}{c^2}\frac{x^2}{\sigma_1^2}\right) \quad \text{.....(xxxii),}$$

or $$\sigma^2_{y_x} = \sigma_2^2(1-r^2)\left\{1+\frac{1}{2n+3}\left(1-\frac{x^2}{\sigma_1^2}\right)\right\} \quad \text{.....(xxxii)}\textit{bis.}$$

Similarly $$\sigma^2_{x_y} = \sigma_1^2(1-r^2)\left\{1+\frac{1}{2n+3}\left(1-\frac{y^2}{\sigma_2^2}\right)\right\} \quad \text{.....(xxxiii).}$$

(xxxii) indicates that the maximum value of the standard deviation of an array of y's is $\sigma_2\sqrt{1-r^2}\sqrt{1+\frac{1}{2n+3}}$, and that this is central, and the minimum value is zero and reached at the boundaries $x = \pm\sqrt{2(n+2)}\sigma_1$ of the frequency surface. If n be a large positive quantity the latter values will be beyond the limits of any moderate sized sample and the term $\frac{1}{2n+3}$ will be very small, or the surface will be very nearly homoscedastic. On the other hand for n small, of order 2 or 3, the tendency will be for very small values of the variation of y to occur with the very large positive or negative values of x. Thus the present surface corresponds to those cases in which there is almost perfect association of one variate with the second for extreme values of the second. The scedastic curve is now an ellipse, with its centre at the mean, and axes those of the variates with magnitudes

$$2\sqrt{\frac{2(n+2)}{2n+3}}\sigma_2\sqrt{1-r^2} \quad \text{and} \quad 2\sqrt{2(n+2)}\sigma_1$$

for σ_{y_x} and x respectively.

(4) A number of interesting special cases present themselves as β_2 gets smaller.

(i) When $\beta_2 = 2{\cdot}25$, $n = 1$, and we have

$$z = \frac{N}{3\pi\sigma_1\sigma_2\sqrt{1-r^2}}\left\{1-\frac{1}{6(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)\right\} \quad \text{.....(xxxiv).}$$

The frequency surface is now a paraboloid or rather that portion of it above the z-plane. The marginal frequencies are of type

$$\phi_1(x) = \frac{8N}{3\sqrt{6}\pi}\frac{1}{\sigma_1}\left(1-\frac{x^2}{6\sigma_1^2}\right)^{\frac{3}{2}} \quad \text{.....(xxxv),}$$

and present no special feature.

(ii) If n now pass through all the positive fractional values, we may note that when $n = \frac{1}{2}$*, the frequency surface is the upper half of an ellipsoid, and the arrays are accordingly ellipses. The marginal frequencies are parabolas.

* $\beta_2 = 1\frac{1}{2} = 2{\cdot}1429$ nearly.

(iii) If $\beta_2 = 2$, the frequency surface becomes an elliptic cylinder, all values of x and y within a given area being equally probable. In this case both the marginal frequencies and the scedastic curves are portions of ellipses and given by the types

$$\phi_1(x) = \frac{N}{\pi\sigma_1}\left(1 - \frac{x^2}{4\sigma_1^2}\right)^{\frac{1}{2}} \quad \text{.....(xxxvi),}$$

$$\sigma^2_{y_x} = \sigma_2^2(1-r^2)\left\{1 + \frac{1}{3}\left(1 - \frac{x^2}{\sigma_1^2}\right)\right\} \quad \text{.....(xxxvii).}$$

(iv) When β_2 becomes less than 2, n becomes negative, and the frequency surface involves no contradictions up to $n = -\frac{1}{2}$, or $\beta_2 = 1{\cdot}8$. The marginal totals are now the rectangles

$$\phi_1(x) = \frac{N}{\sigma_1}\frac{1}{2\sqrt{3}} \text{ from } x = -\sqrt{3}\sigma_1 \text{ to } +\sqrt{3}\sigma_1,$$

$$\phi_2(y) = \frac{N}{\sigma_2}\frac{1}{2\sqrt{3}} \text{ from } y = -\sqrt{3}\sigma_2 \text{ to } +\sqrt{3}\sigma_2 \quad \text{.....(xxxviii).}$$

The scedastic curves are still ellipses, i.e.

$$\left.\begin{aligned} \sigma^2_{y_x} &= \sigma_2^2(1-r^2)\left\{1 + \frac{1}{2}\left(1 - \frac{y^2}{\sigma_2^2}\right)\right\} \\ \sigma^2_{x_y} &= \sigma_1^2(1-r^2)\left\{1 + \frac{1}{2}\left(1 - \frac{x^2}{\sigma_1^2}\right)\right\} \end{aligned}\right\} \quad \text{.....(xxxix),}$$

and the surface is

$$z = \frac{N}{6\pi\sigma_1\sigma_2\sqrt{1-r^2}} \frac{1}{\left\{1 - \frac{1}{3}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right\}^{\frac{1}{2}}} \quad \text{.....(xl).}$$

Up to $n = -\frac{1}{2}$ from $n = 0$, the surface is cup-shaped asymptoting to the elliptic cylinder

$$2(n+2)(1-r^2) = \frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2},$$

or, we have reached U-shaped frequency surfaces.

(v) Beyond $n = -\frac{1}{2}$, i.e. for lower values of β_2 than 1·8, the marginal totals also become U-shaped curves, and the system reaches a limit when $n = -1$ or $\beta_2 = 1{\cdot}5$. In this case by (xxvi) $z_0 = 0$. Therefore by (xxix) z will be zero for all values of x and y except those along the boundary

$$\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} = 2(1-r^2).$$

Accordingly all the x-arrays (and y-arrays) consist of two concentrated lumps placed on this curve, i.e. we have an elliptic ring of frequency; for any given value of x alternative values of y only are possible. The marginal frequencies are, however, continuous, being given by

$$\left.\begin{aligned} \phi_1(x) &= \frac{N}{\pi}\frac{1}{\sqrt{2\sigma_1^2 - x^2}} \\ \phi_2(y) &= \frac{N}{\pi}\frac{1}{\sqrt{2\sigma_2^2 - y^2}} \end{aligned}\right\} \quad \text{.....(xli),}$$

and the scedastic curves by

$$\left.\begin{aligned}\sigma^2_{y_x} &= \sigma_2^2(1-r^2)\left(2-\frac{x^2}{\sigma_1^2}\right)\\ \sigma^2_{x_y} &= \sigma_1^2(1-r^2)\left(2-\frac{y^2}{\sigma_2^2}\right)\end{aligned}\right\} \quad \text{.....(xlii).}$$

Beyond $n=-1$ it does not seem possible to follow this class of frequency surface further. It may be very doubtful indeed whether even this ring of frequency corresponds to any statistical experience. But there is much that is suggestive even in these non-skew regression surfaces for the more general theory of bi-variate frequency surfaces. Such matters are the limited and unlimited bases and the appearance of cup-shaped surfaces of frequency*.

(5) Let us consider lastly formulae for the volume of such surfaces outside a given contour. This may be defined by ψ, where

$$\psi^2 = \frac{1}{2(n-2)}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)$$

in the case of (xix).

The area of the ellipse inside ψ is πAB, where A and B are the semi-axes; it is easily found that

$$2\pi(n-2)\sqrt{1-r^2}\sigma_1\sigma_2\psi^2$$

or the elementary area of the annulus between ψ and $\psi+\delta\psi$, is

$$4\pi(n-2)\sqrt{1-r^2}\sigma_1\sigma_2\psi\delta\psi.$$

Consequently the volume from $\psi=\psi$ to $\psi=\infty$ is

$$\int_\psi^\infty z\times 4\pi(n-2)\sqrt{1-r^2}\sigma_1\sigma_2\psi\delta\psi = N(n-1)\int_\psi^\infty \frac{2\psi\,d\psi}{(1+\psi^2)^n} = \frac{N}{(1+\psi^2)^{n-1}}.$$

Consequently the chance of an individual occurring outside the contour ψ is

$$\frac{1}{(1+\psi^2)^{n-1}} \quad \text{.....(xliii).}$$

This gives a very simple expression for the chance of the occurrence of individuals in the population given by (xvi) who are more improbable than an observed individual.

Half the volume of the frequency surface will lie inside the ellipse defined by

$$\psi_0 = \{2^{\frac{1}{n-1}}-1\}^{\frac{1}{2}} \quad \text{.....(xliv).}$$

For the surface given by (xxix) we must take

$$\psi^2 = \frac{1}{2(n+2)}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right).$$

The contour element of area will be

$$4\pi(n+2)\sqrt{1-r^2}\sigma_1\sigma_2\psi\,d\psi,$$

* The correlation surface of the extents of cloudiness at two meteorological stations would certainly be cup-shaped.

and the volume of the solid between the contour ψ and 1 is

$$N\int_{\psi}^{1}(n+1)(1-\psi^2)^n\,2\psi\,d\psi = N(1-\psi^2)^{n+1}.$$

Accordingly the chance of an individual occurring outside ψ is

$$(1-\psi^2)^{n+1} \quad \text{.....(xlv).}$$

Both (xliii) and (xlv) are very readily calculated for given values of ψ and n. The "probability" ellipse for the latter case is given by

$$\psi = \left(1 - \frac{1}{2^{\frac{1}{n+1}}}\right)^{\frac{1}{2}},$$

and is again easily determined when n is known.

While these surfaces are of far greater generality than the normal surface of frequency, the writer recognises that they form only a very small section of the whole range of frequency surfaces, even if we confine that range to cases in which β_1, β_2, β_1', β_2'—without regard to higher β's—are supposed to be arbitrary for both variables. Such surfaces are, it has been known for a long time, incompatible with linear regression and homoscedasticity. Professor Narumi* has recently approached the subject from another aspect, namely by inquiring what types of regression can be associated with particular types of scedasticity. The extent of generality of the present skew surface is more exactly defined by his work†.

Considerable progress would be made were it feasible to discover a surface for which r, σ_1, σ_2, β_1, β_1', β_2, β_2' were arbitrary. The discovery of a surface in which the higher β's also are arbitrary (or at least some of them) is really not of very great importance considering the average size of statistical samples, as the probable errors of these higher β's are well able to cover most divergencies.

* *Biometrika*, Vol. xv. pp. 77—88 and pp. 209—221.
† *loc. cit.* pp. 217—221.

ON FURTHER INEQUALITIES WITH POSSIBLE APPLICATION TO PROBLEMS IN THE THEORY OF PROBABILITY.

By Professor SEIMATSU NARUMI.

1. Let $\phi(x)$ be any law of frequency, its range of distribution being $-\infty$ and $+\infty$, and

$$\int_{-\infty}^{\infty} \phi(x)\,dx = \int_{0}^{\infty} \{\phi(x) + \phi(-x)\}\,dx = 1.$$

If we write

$$m^n = \int_0^{\infty} x^n \{\phi(x) + \phi(-x)\}\,dx, \quad n > 0;$$

and

$$y = \int_0^{xm} \{\phi(x) + \phi(-x)\}\,dx,$$

then the inverse function of the last relation may be written as

$$z = \psi(y),$$

where, since $\phi(x) + \phi(-x)$ is always positive, at least, non-negative, the function $\psi(y)$ is a non-negative, non-decreasing function; and further we can readily find the following two properties

$$\int_0^1 \psi(y)^n\,dy = 1 \quad \text{.....(A)},$$

$$\psi(0) = 0 \quad \text{.....(B)}.$$

The object of this paper is to obtain certain inequalities which may be called the generalized Tchebycheff's inequalities and the generalized Gauss' inequalities. The results here obtained may not be serviceable for practical purposes, but they have some theoretical interest.

2. *The General Case.* Let us suppose that the curve $z = \psi(y)$ passes through a point (y_0, z_0), that is,

$$z_0 = \psi(y_0).$$

Then clearly we shall have

$$\psi(y) \geqq 0, \quad \text{when} \quad 0 \leqq y \leqq y_0,$$

or

$$\int_0^{y_0} \psi(y)^n\,dy \geqq 0;$$

and

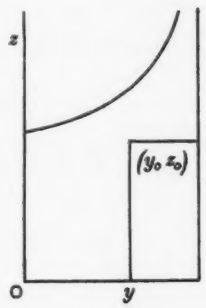
$$\psi(y) \geqq z_0, \quad \text{when} \quad y_0 \leqq y \leqq 1,$$

or

$$\int_{y_0}^1 \psi(y)^n\,dy \geqq z_0^n (1 - y_0).$$

Therefore, by the condition (A), we have

$$1 \geqq z_0^n (1 - y_0).$$



Evidently the equality will occur when and only when

$$\psi(y) = 0, \quad \text{when} \quad 0 \leqq y \leqq y_0,$$
$$\psi(y) = z_0, \quad \text{when} \quad y_0 \leqq y \leqq 1.$$

Now, dropping the subscripts of y_0 and z_0, we have

$$z^n(1-y) \leqq 1$$

or
$$1 \geqq y \geqq \left(1 - \frac{1}{z^n}\right), \quad \text{when} \quad z \geqq 1,$$

which is nothing but the generalized Tchebycheff's inequality due to Professor Karl Pearson*. Professor Pearson supposed n to be an even number, and Mr P. Cantelli† n to be an integer. It is to be noted here that, in obtaining this result, no assumption with regard to the nature of the positive constant n has been used so long as the integral

$$\int_0^\infty x^n \{\phi(x) + \phi(-x)\}\, dx$$

exists. The equality occurs if we take the origin at one lump when the law of distribution is that of two lumps. If we confine ourselves to the case when the law of frequency is continuous, the equality does not occur, but we can easily find the continuous function $\phi(x)$ such that the difference of both sides of the above inequality is less than any positive number, however small it may be.

A similar statement may be made for every inequality in this paper. I shall not repeat it. It is not difficult to find the case in which the equality arises. The law of the frequency distribution may be a rectangle, or it may be needful to add a lump to the rectangular frequency for the equality to occur.

3. *The* U*-shaped or* J*-shaped Curve.* To obtain a closer inequality, we must assume some further limitation of the law of the frequency distribution. We shall now assume that the function $\phi(x) + \phi(-x)$ is increasing, at least non-decreasing, in the interval $0 \leqq x \leqq bm$, $b > 1$; and put

$$a = \int_0^{bm} \{\phi(x) + \phi(-x)\}\, dx.$$

Then we easily find that $\psi(y)$ is a non-negative, concave function of y in the interval

$$0 \leqq y \leqq a \leqq 1.$$

Case (i). $1 < b < (1+n)^{\frac{1}{n}}$. In this case we shall have

$$1^\circ. \quad \frac{z}{b} \geqq y \geqq 0, \qquad \text{when} \quad b_1 \geqq z \geqq 0,$$

$$2^\circ. \quad \frac{z}{b} \geqq y \geqq \frac{z}{b} - \frac{b-z}{b^n - z^n}\,\frac{n+1-b^n}{b}, \quad \text{when} \quad b \geqq z \geqq b_1,$$

$$3^\circ. \quad 1 \geqq y \geqq 1 - \frac{1 - \dfrac{b^n}{n+1}}{z^n - \dfrac{b^n}{n+1}}, \qquad \text{when} \quad z \geqq b;$$

* *Biometrika*, Vol. XII. pp. 284—298. † See *Comptes Rendus*, Tom. 175, p. 1382.

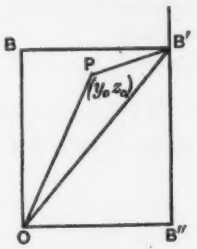
where b_1 is the positive root of the equation

$$z\left(\frac{b^n - z^n}{b - z}\right) = n + 1 - b^n.$$

Consider a special case in which $z = b$. Then we have

$$1 \geqq y \geqq \left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{z^n}\right), \text{ when } z > 1,$$

under the hypothesis that the function $\phi(x) + \phi(-x)$ is non-decreasing in the interval $0 \leqq x \leqq zm$. This inequality is quite analogous to that of Mr B. Meidell*. In his case the function $\phi(x) + \phi(-x)$ is non-increasing in the interval $0 \leqq x \leqq zm$.



Draw a straight line to which the equation is $z = b$, to meet at the points B and B' the z-axes and $y = 1$ respectively.

Let us suppose that the curve $z = \psi(y)$ passes through a point P whose co-ordinates are y_0, z_0 respectively, when

$$0 \leqq z_0 \leqq b.$$

Join the point P to the origin O and to the point B'. Since $z = \psi(y)$ is concave when $0 \leqq y \leqq y_0$, it cannot meet the line OP when $0 < y < y_0$, because two points O and P are common, therefore

$$\psi(y) \geqq \frac{z_0}{y_0} y, \text{ when } 0 \leqq y \leqq y_0,$$

or

$$\int_0^{y_0} \psi(y)^n \, dy \geqq \frac{y_0 z_0^n}{n+1}.$$

If the curve $z = \psi(y)$ meets the line PB' when $y_0 < y < 1$, it must meet the line $y = 1$ when $z_0 < z < b$ which contradicts the hypothesis that the function $\phi(x) + \phi(-x)$ is non-decreasing in the interval $0 \leqq x \leqq bm$. Therefore the curve $z = \psi(y)$ cannot meet the line PB' when $y_0 < y < 1$; from which we find

$$\psi(y) \geqq z_0 + \frac{b - z_0}{1 - y_0}(y - y_0), \text{ when } y_0 \leqq y \leqq 1,$$

or

$$\int_{y_0}^1 \psi(y)^n \, dy \geqq \frac{1 - y_0}{n+1} \frac{b^{n+1} - z_0^{n+1}}{b - z_0}.$$

Consequently

$$\int_0^1 \psi(y)^n \, dy \geqq \frac{y_0 z_0^n}{n+1} + \frac{1 - y_0}{n+1} \frac{b^{n+1} - z_0^{n+1}}{b - z_0},$$

and, by the condition (A) we have

$$1 \geqq \frac{y_0 z_0^n}{n+1} + \frac{1 - y_0}{n+1} \frac{b^{n+1} - z_0^{n+1}}{b - z_0},$$

or, dropping the subscripts,

$$y \geqq \frac{z}{b} - \frac{b - z}{b^n - z^n} \frac{n + 1 - b^n}{b},$$

* *Comptes Rendus*, Tom. 175, 1922, p. 892.

when $b_1 \leqq z \leqq b$; and b_1 is the positive root of the equation

$$\frac{z}{b} = \frac{b-z}{b^n - z^n} \frac{n+1-b^n}{b},$$

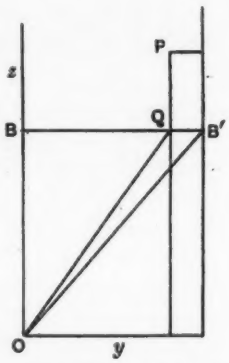
i.e.
$$\frac{z(b^n - z^n)}{b-z} + b^n = n+1.$$

Therefore the lower limit of y has been established when $0 \leqq z \leqq b$. Now the upper limit of y is easily obtained. For if the curve $z = \psi(y)$ meets the line OB', it must meet the line $y=1$ when $0 < z < b$, which contradicts the hypothesis that the function $\phi(x) + \phi(-x)$ is non-decreasing in the interval $0 \leqq x \leqq bm$. Therefore we have

$$\frac{z}{b} \geqq y, \text{ when } 0 \leqq z \leqq b.$$

To prove 3° let us consider a point P whose co-ordinates are y_0, z_0 and

$$\psi(y_0) = z_0 > b.$$



Draw a perpendicular, passing through P, to the y-axes, and let it meet the line BB', at a point Q, and join Q to the origin.

The curve $z = \psi(y)$ cannot lie entirely below the line OQ when y lies in the interval 0 to y_0. For if it were so, the curve $z = \psi(y)$ would meet the line BB' at a point which is an interior point of the segment QB', and therefore cannot pass through the point P. Now the curve $z = \psi(y)$ cannot meet the line OQ when y lies in the interval 0 to y_0. For if it were so, the curve $z = \psi(y)$ would meet the line OQ at two points, and therefore it cannot meet again, so that the curve $z = \psi(y)$ will meet the segment QB' at a point which is an interior point of QB'. This is impossible. Consequently the curve $z = \psi(y)$ must lie entirely above the line OQ, that is to say,

$$\psi(y) \geqq \frac{b}{y_0} y, \text{ when } 0 \leqq y \leqq y_0,$$

or
$$\int_0^{y_0} \psi(y)^n dy \geqq \frac{b^n y_0}{n+1}.$$

It is clear that
$$\psi(y) \geqq z_0, \text{ when } y_0 \leqq y \leqq 1,$$

or
$$\int_{y_0}^1 \psi(y)^n dy \geqq (1-y_0) z_0^n.$$

Therefore
$$\int_0^1 \psi(y)^n dy \geqq \frac{b^n y_0}{n+1} + (1-y_0) z_0^n, \text{ when } z_0 > b.$$

From the condition (A)
$$1 \geqq \left(\frac{b^n}{n+1} - z_0^n\right) y_0 + z_0^n,$$

or, dropping the subscripts, we have

$$1 \geqq y \geqq 1 - \frac{1 - \dfrac{b^n}{n+1}}{z^n - \dfrac{b^n}{n+1}}, \quad \text{when} \quad z \geqq b,$$

which is the inequality 3°. It should be noted that this inequality is a little closer than the generalized Tchebycheff's inequality. For we have

$$1 - \frac{1 - \dfrac{b^n}{n+1}}{z^n - \dfrac{b^n}{n+1}} = \left(1 - \frac{1}{z^n}\right)\left(1 + \frac{\dfrac{b^n}{n+1}}{z^n - \dfrac{b^n}{n+1}}\right) > \left(1 - \frac{1}{z^n}\right).$$

Case (ii). $b = (1+n)^{\frac{1}{n}}$.

In this case we shall have

1°. $y = (1+n)^{-\frac{1}{n}} z$, when $0 \leqq z \leqq (1+n)^{\frac{1}{n}}$,

2°. $y = 1$, when $(1+n)^{\frac{1}{n}} \leqq z$.

These relations will be readily obtained by making $\dfrac{n+1}{b^n} \to 1$ in the former case.

Case (iii). $b > (1+n)^{\frac{1}{n}}$.

This case is clearly impossible.

4. *The "Cocked-Hat" Shaped Curves.* Finally, let us consider the case in which the function $\phi(x) + \phi(-x)$ is steadily decreasing, at least non-increasing, in the interval $0 \leqq x \leqq bm$, $b > 1$. Evidently the function $\psi(y)$ is a non-negative, convex function of y when $0 \leqq y \leqq a$, where a is given by

$$a = \int_0^{bm} \{\phi(x) + \phi(-x)\}\, dx.$$

We shall now prove that

1°. $1 \geqq y \geqq \dfrac{n+1}{nb}\left(1 - \dfrac{1}{b^n}\right) z$, when $0 \leqq z \leqq \dfrac{nb}{n+1}$;

2°. $1 \geqq y \geqq 1 - \dfrac{1}{b^n}$, when $\dfrac{nb}{n+1} \leqq z \leqq b$;

3°. $1 \geqq y \geqq 1 - \dfrac{1}{z^n}$, when $b \leqq z$.

These inequalities may be termed generalized forms of Mr B. Meidell's* inequality; and his inequality corresponds to the special case in which

$$z = \frac{n}{n+1} b$$

under the same hypothesis.

* *Loc. cit.* p. 246, above.

To prove the first and the second inequalities, it is sufficient to show that for any function $\psi(y)$ we have

$$\psi(y) \leqq \frac{nb}{n+1}\frac{b^n}{b^n-1}y, \text{ when } 0 \leqq y \leqq 1-\frac{1}{b^n},$$

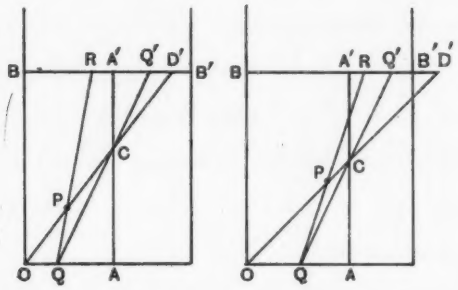
provided that the function $\phi(x)+\phi(-x)$ is non-increasing in the interval

$$0 \leqq x \leqq bm.$$

The third inequality is merely a repetition of that in our Section 1.

Let a point A' be the intersection of the curve $(1-y)z^n=1$ and the line $z=b$. Draw a perpendicular AA' to the y-axes. Find a point C on the line AA' such that

$$AC : AA' = n : n+1.$$



Now join the point C to the origin, and produce it to meet at a point D' the line BB'. Then we know that the integral $\int_0^1 z^n\,dy$ along the broken line $OCD'B'$, i.e.

$$\int_0^{\frac{n+1}{n}\left(1-\frac{1}{b^n}\right)} \left(\frac{nb}{n+1}\frac{b^n}{b^n-1}y\right)^n dy + \int_{\frac{n+1}{n}\left(1-\frac{1}{b^n}\right)}^1 b^n\,dy,$$

is equal to 1. Draw a tangent OT to the curve $z=\psi(y)$ at the origin, and let it meet the line BB' at a point T. Evidently

$$1 = \int_0^1 \psi(y)^n\,dy > \int_0^1 z^n\,dy \text{ along the broken line } OTB'.$$

Hence the point T on the line $A'B'$ cannot lie on the segment $A'D'$. Consequently, in the vicinity of the origin, we have

$$\psi(y) \leqq \frac{nb}{n+1}\frac{b^n}{b^n-1}y.$$

Now suppose that the curve $z=\psi(y)$ cut the line OC at a point P which lies on the segment OC. Draw a tangent at the point P to the curve $z=\psi(y)$, which meets at a point Q the y-axes, and at a point R the line BB'.

We have now

$$\int_0^1 \psi(y)^n\,dy > \int_0^1 z^n\,dy \text{ along the broken line } OQPRB';$$

and

$$\int_0^1 z^n\,dy \text{ along the broken line } OQPRB' > \int_0^1 z^n\,dy \text{ along the broken line } OQCQ'B'.$$

But evidently we have

$$\int_0^1 z^n\,dy \text{ along the broken line } OQCQ'B' = 1,$$

and therefore

$$\int_0^1 \psi(y)^n\,dy > 1,$$

which contradicts the condition (A) of p. 245.

Consequently we see that the curve $z=\psi(y)$ cannot cut the segment OC, that is to say

$$\psi(y) \leqq \frac{nb}{n+1}\frac{b^n}{b^n-1}y,$$

for all values of y in the interval $0 \leqq y \leqq \frac{b^n}{b^n-1}$. Therefore the first and the second inequalities have been established.

Let us now suppose that the function $\phi(x)+\phi(-x)$ is non-increasing in the whole interval of x. This corresponds to $b=\infty$. In this case we see that the largest part of the possible domain of y for all values of b is the required domain of y, i.e.

$$1^\circ. \quad 1 \geqq y \geqq (1+n)^{-\frac{1}{n}}z, \qquad \text{when } 0 \leqq z \leqq 1;$$

$$2^\circ. \quad 1 \geqq y \geqq 1-\left(\frac{n}{n+1}\right)^n\frac{1}{z^n}, \quad \text{when } 1 \leqq z.$$

These inequalities may be called generalized forms of Gauss' inequalities*.

* See F. Bernstein and M. Krafft: "Integralungleichungen konvexer Funktionen," *Göttingen Nachrichten*, 1914, S. 299.

[The argument is as follows:

Returning to the inequalities on p. 249:

$$(1) \quad 1 \geqq y \geqq \frac{n+1}{nb}\left(1-\frac{1}{b^n}\right)z, \text{ when } 0 \leqq z \leqq \frac{nb}{n+1},$$

$$(2) \quad 1 \geqq y \geqq 1-\frac{1}{b^n}, \text{ when } \frac{nb}{n+1} \leqq z \leqq b,$$

we note that (1) is true for all values of $b>1$. Now y from its definition is really independent of b, so that y will be greater than or equal to the maximum value of $\frac{n+1}{nb}\left(1-\frac{1}{b^n}\right)z$, provided that the upper limit of z is given its corresponding value. The maximum value of $\frac{n+1}{nb}\left(1-\frac{1}{b^n}\right)$ is $(1+n)^{-\frac{1}{n}}$ and the corresponding value of b is $(1+n)^{\frac{1}{n}}$.

(1) accordingly becomes

$$(1)^{bis} \quad 1 \geqq y \geqq (1+n)^{-\frac{1}{n}}z, \text{ when } 0 \leqq z \leqq \frac{n}{n+1}(1+n)^{\frac{1}{n}}.$$

5. *Remarks on Professor B. H. Camp's Inequality**.

In the generalized Tchebycheff's inequality due to Professor K. Pearson, the equality may occur in the case of "three lumps" which may not arise in practice but are possible theoretically. In mathematical language this may be stated as follows: For any values of s and λ, we can find a positive, integrable function $f(x)$ such that the difference between $\frac{\beta_{2s-2}}{\lambda^{2s}}$ and $P(\lambda\sigma)$ is less than any positive number ϵ however small ϵ may be. Consequently we cannot replace $\frac{\beta_{2s-2}}{\lambda^{2s}}$ by any lower limit, unless we assume some other properties of the function $f(x)$. It is in this direction that Professor Camp has moved. He assumes that

$$\frac{f'(x)}{x} \leqq 0 \text{ for } |x| \geqq c\sigma, \quad c \geqq 0;$$

This reduces to the value in the text as upper limit for z, i.e. unity if $1 \leqq \frac{n}{n+1}(1+n)^{\frac{1}{n}}$ which is the case if $n \geqq 1$. If $0<n<1$ we must retain the limit here given, not the one in the text.

Similarly for (2) it is necessary to find the maximum value of $1-\frac{1}{b^n}$ under the condition that $\frac{nb}{n+1} \leqq z$. The greatest value of b will be $\frac{n+1}{n}z$ and so the least value of $\frac{1}{b^n}$ will be $\left(\frac{n}{n+1}\right)^n \frac{1}{z^n}$. The maximum value of $1-\frac{1}{b^n}$ is accordingly $1-\left(\frac{n}{n+1}\right)^n \frac{1}{z^n}$.

(2) then becomes

$$(2)^{bis} \quad 1 \geqq y \geqq 1-\left(\frac{n}{n+1}\right)^n \frac{1}{zn}, \text{ when } \frac{n}{n+1}(1+n)^{\frac{1}{n}} \leqq z.$$

In the text the limit of z is given as 1, but this inequality will certainly be true if

$$(1+n)^{-\frac{1}{n}} z \geqq 1-\left(\frac{n}{n+1}\right)^n \frac{1}{z^n}$$

for values of z between 1 and $\frac{n}{n+1}(1+n)^{\frac{1}{n}}$.

As z increases these functions increase; their values when $z=1$ are $(1+n)^{-\frac{1}{n}}$ and $1-\left(\frac{n}{n+1}\right)^n$ respectively; when $z=\frac{n}{n+1}(1+n)^{\frac{1}{n}}$, they are $\frac{n}{n+1}$ and $\frac{n}{n+1}$. To justify the limit taken in the text, we must show that

$$(1+n)^{-\frac{1}{n}} \geqq 1-\left(\frac{n}{n+1}\right)^n.$$

This is true because

$$(1+n)^{-\frac{1}{n}} + \left(\frac{n}{n+1}\right)^n \geqq 1$$

for all values of n.

Gauss' principle is obtained from $(1)^{bis}$ and $(2)^{bis}$ by putting $n=2$, i.e.

$$(1)^{ter} \quad 1 \geqq y \geqq \frac{1}{\sqrt{3}} z, \text{ when } 0 \leqq z \leqq \frac{2}{\sqrt{3}}.$$

$$(2)^{ter} \quad 1 \geqq y \geqq 1-\frac{4}{9}\frac{1}{z^2}, \text{ when } \frac{2}{\sqrt{3}} \leqq z.$$

See Gauss, *Theoria combinationis observationum erroribus minimis obnoxiae*, *Werke*, Bd. IV. S. 11. ED.]

* "A new Generalization of Tchebycheff's Statistical Inequality," *Bulletin of the American Mathematical Society*, Vol. XXVIII. pp. 427—432.

and reaches a new generalized inequality

$$P(\lambda\sigma) \leqq \frac{\beta_{2s-2}}{\lambda^{2s}} \frac{\left(\frac{2s}{2s+1}\right)^{2s}}{1+\phi} + \frac{\phi}{1+\phi} P(cs),$$

where

$$\phi = \frac{\left(\frac{c}{\lambda}\frac{2s}{2s+1}\right)^{2s}}{(2s+1)\left(\frac{c}{\lambda}-1\right)}.$$

This result may be serviceable in many cases in practice.

Professor Camp does not state whether the equality can or cannot occur, but I think this matter is important from the theoretical point of view. To prove the inequality he starts with a lemma which is very interesting and seems to me the essential point of the paper. Unfortunately the proof of the lemma is not quite rigorous, although a rigorous proof can be obtained by a slight modification. The theorem is merely an application of the lemma.

The reader must be careful when he uses the tables in this paper, since a few misprints appear in every table*.

[* Professor Camp has kindly provided a supplementary note to his original paper in the *Bulletin of the American Mathematical Society* which is published in the *Miscellanea* of our present number. ED.]

BIOMETRICAL STUDIES ON THE ANTHROPOLOGY OF RUSSIA.

By ETHYME TSCHEPOURKOWSKY.

Applying the geographical method to measurements of nearly 50,000 Russian peasants* I was led to the chief types given in the map, Plate I.

(i) There exists in the East of European Russia (Riazan, Tambof and Penza governments)† an island of relative dolichocephaly, namely 16 contiguous districts, the mean cephalic index of each not exceeding 79·9. Such a low mean index is nowhere else to be met with. The empirical distribution of those districts is provided in the accompanying Table I (7816 men). The hair- and eye-colour in this region is relatively dark.

(ii) In the other parts of Great Russia the watersheds are inhabited by a much more brachycephalic type: the mean index of districts is there nearly 83—87. This type is relatively blond. The same population with much more accentuated blondness and brachycephaly occupies the swamps of White Russia along the rivers Pripiet and Berezina, affluents of the river Dniepr. The river valleys in Great Russia are occupied by an intermediate or mixed type of index: 81—82.

(iii) The Little Russian population is also brachycephalic but very dark.

(iv) Along the valleys of the Nieman and West Dwina (w. Duna) and in government Moghileff there are remnants of the Scandinavian type, blond and relatively dolichocephalic (80—81).

According to considerations I will not reproduce here‡ I think that the first of these types (East Russian) is the remnant of an ancient population of Turco-Finnish and partly slavic origin; the second invaded Great Russia much later (700—1000 years ago) from the swamps of White Russia and has an affinity with the round-barrow population of Britain§; the third colonized Little Russia quite recently from the Carpathians (*Homo alpinus*) after the Tartar invasion.

I have used biometric methods in the following cases:

(1) To determine if the number of observations was sufficient to calculate the mean index of districts or subdistricts (communities) I calculated the coefficient

* Nearly 40,000 of which were measured by myself and the rest by my student pupils; a small number was extracted out of various works dealing with special investigations.

† Russia is administratively divided into Governments (Goubernia), those into Districts (Ouïezd), and the last into Subdistricts or Communities (Volost).

‡ For the reasons which compelled me to choose cephalic index as my working character see "Anthropologische Studien" (*Archiv für Anthropologie*, 1912), and two dissertations published in Russian.

§ See A. Keith, "The Bronze Age Invaders of Britain." *J. of R. A. I.*, 1915.

TABLE I. *The Curves of Cephalic Index (Men) according to Regions.*

Index	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97
1. The East Region	—	3	9	11	42	92	195	368	474	786	862	*1165*	1011	875	655	461	365	188	95	36	19	14	5	3	1	—	—	—	—	—
2. The West of the Eastern Region	1	—	3	4	3	15	39	68	83	162	202	*314*	306	304	227	241	166	102	57	38	16	19	—	2	1	2	—	—	—	—
3. The Intermediate Zone	—	—	—	—	4	9	6	18	35	52	65	109	120	*151*	123	133	90	88	48	39	22	18	—	2	—	1	—	—	—	—
4. The East of the Waldaï Region	—	—	—	—	—	2	3	7	10	17	29	69	91	131	109	114	*153*	106	86	60	49	14	13	7	—	4	2	—	—	—
5. The Centre of the Waldaï Region	—	—	—	—	—	—	—	—	2	—	3	8	6	11	11	18	15	*23*	10	10	9	5	1	2	—	—	—	—	—	—
6. The Mordwa-moksha Tribe	—	—	—	—	3	1	5	14	22	36	*55*	44	34	28	20	14	6	4	2	1	1	—	—	—	—	—	—	—	—	—

					1st Component			2nd Component		
	N	Σ	M	κ	n_1	σ_1	m_1	n_2	σ_2	m_2
1. The East Region	7815	2·98	79·46	0·0257	6505 (83·2)	2·78	79·27	1310 (16·8)	3·58	30·41
2. The West of the Eastern Region	2373	3·16	80·55	0·0072	—	—	—	—	—	—
3. The Intermediate Zone	1133	3·30	81·57	0·0033	815 (71·9)	3·12	83·57	318 (28·1)	2·85	80·78
4. The East of the Waldaï Region	1076	3·20	83·07	0·0114	928 (36·2)	3·02	82·92	148 (13·8)	3·97	83·92
5. The Centre of the Waldaï Region	134	3·05	83·87	−0·0241	—	—	—	—	—	—
6. The Mordwa-moksha Tribe	290	2·64	78·96	0·1624	—	—	—	—	—	—

n_1, N = Number of Observations. σ_1, Σ = Standard Deviation. m_1, M = Mean. κ = Criterion.

of correlation: (*a*) between two mean indices for the same districts deduced from two series of observations: by myself, or by myself and another author. It was $0{\cdot}912 \pm {\cdot}022$; (*b*) between the lowest number of observations in each district and the above mentioned differences of the mean indices, deduced from two series of my own observations. The result was negative ($r = -0{\cdot}195 \pm {\cdot}077$), i.e. with the increase of the number of observations the differences decrease (see Tables II and III).

TABLE II.

The Mean Cephalic Index of Districts.

Mine. \ Other Authors.	*78*	*79*	*80*	*81*	*82*	*83*	*84*	Totals
78	—	—	—	—	—	—	—	—
79	1	3	1	1	—	—	—	6
80	—	—	1	1	—	—	—	2
81	—	—	1	3	—	1	—	5
82	—	—	—	2	1	—	—	3
83	—	—	—	—	3	8	—	11
84	—	—	—	—	—	—	1	1
Totals	1	3	3	7	4	9	1	28

TABLE III.

Correlation between (a) the Differences of Cephalic Index Deduced from Two Series of Observations; and (b) the Number of Cases (Least of Two).

Number of Cases. \ Differences.	*0·0*	*0·1*	*0·2*	*0·3*	*0·4*	*0·5*	*0·6*	*0·7*	*0·8*	*0·9*	*1·0*	*1·1*	*1·2*	*1·3*	*1·4*	*1·5*	*2·0*	Totals
15—24	—	3	1	1	1	2	1	1	2	—	3	—	2	1	—	1	1	20
25—49	2	4	4	1	4	3	5	1	—	2	2	2	1	1	1	1	—	34
50—74	—	1	—	1	1	—	1	2	3	—	—	—	—	—	—	—	—	9
75—99	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	2
125—150	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
150—175	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
175—200	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
200—225	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
225—250	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
250—275	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
275—300	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	2	9	7	3	8	6	8	5	5	2	5	2	3	2	1	2	1	71
	11		10		14		13		7		7		5		3		1	

(2) To determine if my appreciation of hair-colour was sufficiently exact, I calculated the correlation for the relative number in percentages of extreme blonds and extreme dark-haired in each government (Table IV) and found

$$r = -{\cdot}883 \pm {\cdot}030.$$

TABLE IV.

Correlation between the Percentages of Blonds and Brunettes in Different Governments.

°/₀	—	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Totals
30—35	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
35—40	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
40—45	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	2
45—50	—	—	—	—	—	1	1	—	1	—	—	—	—	—	—	—	—	3
50—55	—	—	—	—	3	—	3	2	—	—	—	—	—	—	—	—	—	8
55—60	—	—	1	1	2	1	—	—	—	—	—	—	—	—	—	—	—	5
60—65	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	2
65—70	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	2	—	1	2	5	2	4	3	1	—	—	—	—	1	1	—	1	23

(3) To solve the question if in the Russian peasant population the hair- and eye-colours are correlated with the cephalic index I observed the colour of 27,000 peasants (soldiers) in the same conditions of light, divided them into geographical groups (districts), in every group calculated the mean index for

TABLE V.

Differences of Cephalic Index of Blonds and Brunettes According to Sign.*

Differences	Types. I_1 −	III_3 +	I −	III +	1 −	3 +	Totals
0·0	2		2		3		7
0·1	2	1	3	4	2	5	17
0·2	1	1	6	5	1	3	17
0·3	2	1	—	5	4	—	12
0·4	2	2	1	1	2	—	8
0·5	4	—	4	—	—	4	12
0·6	2	1	1	1	—	2	7
0·7	—	1	—	1	4	2	8
0·8	—	1	—	1	1	2	5
0·9	1	—	—	2	1	—	4
1·0	—	2	1	—	—	—	3
1·1	—	—	—	—	—	—	—
1·2	1	—	—	1	—	—	2
1·3	—	—	—	1	—	—	1
1·4	—	—	1	1	2	—	4
1·5	—	—	—	—	—	—	—
1·6	—	1	—	—	—	—	1
1·7	—	—	—	1	—	—	1
1·8	—	—	—	1	—	—	1
Totals	15	11	17	25	17	18	110
	2		2		3		
Mean differences	0·46	0·43	0·40	0·59	0·48	0·42	0·47
Id. in per cent.	54	39	39	37	45	47	

* When the cephalic index of the fair group is higher the differences are taken as positive. I_1 signifies blond hair and blue eyes; III_3, dark hair and brown eyes; I, all blond haired; III, all dark-haired; 1, all blue-eyed; 3, all brown-eyed.

each subdivision of hair- and eye-colour* and found (Tables V and VI): (*a*) that the differences between those mean indices are distributed according to the laws of probability; and (*b*) that they also diminish with the increase of the number of observations (*r*-negative†). It confirmed my previous assertion that the fundamental types cannot be discovered with certainty by the sole method of establishing correlations among characters‡.

TABLE VI.

Correlation between the Differences of Cephalic Index of Blonds and Brunette Types and the Number of Observations.

Number of Observations (Least of Two).	Differences.							
	15—25	*25—50*	*50—75*	*75—100*	*100—125*	*125—150*	*150—175*	Totals
0·0	2	1	2	2	—	—	—	7
0·1	6	2	8	—	—	1	—	17
0·2	4	3	5	2	2	—	1	17
0·3	2	5	2	2	—	1	—	12
0·4	4	3	—	—	1	—	—	8
0·5	6	4	1	—	—	1	—	12
0·6	3	2	1	—	1	—	—	7
0·7	5	3	—	—	—	—	—	8
0·8	—	2	2	1	—	—	—	5
0·9	2	1	1	—	—	—	—	4
1·0	2	—	—	1	—	—	—	3
1·1	—	—	—	—	—	—	—	—
1·2	1	1	—	—	—	—	—	2
1·3	1	—	—	—	—	—	—	1
1·4	2	2	—	—	—	—	—	4
1·5	—	—	—	—	—	—	—	—
1·6	1	—	—	—	—	—	—	1
1·7	—	—	—	—	—	—	1	1
1·8	1	—	—	—	—	—	—	1
Totals	42	29	22	8	4	3	2	110

(4) Anthropologists often describe a people (in most cases without any attempts to resolve it into physically different components) by a series of arithmetical means

* Categories for hair were: blond, intermediate, dark. Each group was divided according to the eye-colour into three subgroups: blue, intermediate, brown.

[† $r = -\cdot122 \pm \cdot064$, and cannot therefore be considered definitely significant. Ed.]

‡ The reason is that characters primitively united can become separated by heredity; mixtures can arise and the primitive types can vanish very rapidly. See Hans Friedenthal, "Der Grad der Blutsverwandtschaft." *Die Umschau*, 1916, N. 37, and Otto Ammon, "Zur Theorie der reinen Rassentypen." *Zeitschrift für Morphologie und Anthropologie*, 1900, II, but the total relative number with a character not capable of giving intermediate forms (if a character give intermediate forms the means of different regions or districts will differ until a complete mixture is reached) remains in all the population the same. In this case we can characterize the population by the geographical method, i.e. by defining the percentages of different characters in a given area. If a certain character shows a regularity and not chaos in its geographical distribution, i.e. if it occupies a quite definite region with transition-zones, then only is it a typical character and the type is real. I find that this is true of the cephalic index in European Russia.

for different characters. But if we take within the limits of the same race many determinations of a given character (for instance, if we take from a race ten groups of individuals and deduce a mean cephalic index for each group) these determinations will vary according to a probability curve. Each race is characterized by its own probability curve of this kind and we must compare these curves but not the means of characters deduced from one or two observations. In Table VII we see for instance the curves of mean indices of subdistricts (communities) in different districts of Russia and in Table VIII the curves of percentages of blond type in different governments of the Great and Little Russias.

TABLE VII.

Mean Cephalic Index of Communities (Subdistricts) in Different Regions of East Russia.

M Index	*76*	*77*	*78*	*79*	*80*	*81*	*82*	*83*	*84*	*85*	*86*	Totals
The East Region	1	5	31	70	61	20	2	—	—	—	—	190
The West of it	—	—	6	12	43	39	20	6	—	—	—	126
The Government of Smolensk ...	—	—	—	—	6	11	20	10	—	—	—	47
The Government of Ekaterinosslav	—	—	—	—	—	—	1	12	15	—	—	28
The District of Igoumen	—	—	—	—	—	—	—	3	8	1	1	13
The District of Marioupol (Greeks)	—	—	—	—	—	—	—	—	1	5	1	7
Mordwa of Tambof Government .	—	—	2	4	—	1	—	—	—	—	—	7
The Government of Kostroma ...	—	—	—	—	—	8	22	31	20	4	—	85
									Total ...	...		503

TABLE VIII.

Blond Blue-Eyed Type.

Percentages to all population ...	*0–1*	*1–2*	*2–3*	*3–4*	*4–5*	*5–6*	*6–7*	*7–8*	*8–9*
Five Governments of Little Russia	—	1	2	1	1	—	—	—	—
Eight Governments of Great Russia	—	—	—	—	1	1	1	1	1

(5) Variation, calculated as $\sqrt{\frac{\Sigma x^2}{N}}$ and not defined by a simple statement of extreme variates, gives us a criterion of the relative purity—unmixed character or antiquity—of a race. It follows from Table IX that in Great Russia σ attains its minimum value in the three governments occupied by the East Russian type (Riazan, Tambof, Penza). In Table I we see the empirical index-curves of two extreme regions (East and Waldaï) and of the three transition zones between them. The standard deviation takes its lowest values in the centres of both regions and the maximum values in the middle zone (to be compared with the criteria of those curves quoted below).

I calculated also the correlation coefficient between mean indices and σ for subdistricts (communities) of the Tambof government and found r positive, i.e. with

TABLE IX.

Standard Deviation of Cephalic Index in Governments.

Government	σ	N
1. Penza	1·98	1279
2. Tambof	3·05	8173
3. Riazan	3·12	2520
4. Poltawa	3·13	855
5. Moscow	3·17	833
6. Smolensk ...	3·18	2148
7. Ekaterinosslav ...	3·19	1350
8. Nijny-Novgorod .	3·22	830
9. Tver	3·24	681
10. Moghileff ...	3·27	916
11. Orel	3·27	1733
12. Vladimir ...	3·28	950
13. Toula	3·34	562
14. Minsk	3·37	1180
15. Voroneje ...	3·37	756
16. Kursk	3·62	639

TABLE X. *Standard Deviation of Cephalic Index in Subdistricts (Communities), Districts and Governments.*

σ	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	N	Mean σ
Subdistricts :																								
Mordwa-moksha Tribe ...	1	—	—	—	—	—	—	—	—	1	1	1	—	—	—	—	—	—	—	—	—	—	4	2·55
The Government of Tambof	—	—	—	—	—	—	—	11	8	6	10	4	3	6	2	2	—	1	—	—	—	—	53	2·80
Other Governments ...	—	—	—	—	—	—	—	—	—	2	2	2	6	5	5	4	1	1	—	—	—	—	28	3·08
Districts :																								
The Government of Tambof	—	—	—	—	—	—	—	—	—	—	1	4	4	—	—	—	1	—	—	—	—	—	10	2·98
Five districts of other Governments * ...	—	—	—	—	—	—	—	—	—	—	—	—	1	3	—	1	—	—	—	—	—	—	5	3·12
Other districts of different Governments	—	—	—	—	—	—	—	1	—	1	2	5	6	17	2	8	6	4	—	—	—	1	53	3·15
Governments as a whole †	—	—	—	—	—	—	—	—	—	—	—	—	2	—	6	8	5	1	—	—	—	—	22	3·28

* Two districts of the government of Smolensk, one of Minsk, one of Ekaterinosslav, one of Orel. The mean standard deviation of their 22 communities (subdistricts) is 3·10.

† The number of observations is at least 500 in each government.

the increase of index σ also increases. It proves, in my opinion, that in the brachycephalic subdistricts there are more new invaders than in the dolichocephalic*.

Table X shows the variation (σ) in communities (number of observations at least 50 for each), districts (number of observations at least 100) and governments (number of observations at least 500). We see that the lowest variation is appropriated to the Finnish natives of this region—Mordwa. Secondly, the extreme skewness of the σ curve in subdistricts of the government of Tambof is not without a certain interest. This curve begins with the ordinate 2·5, corresponding to the variation of the above-mentioned Finnish people (Mordwa). It proves once more that Mordwa was one of the constituents of the East Russian type. Thirdly, we see that the mode of the curve of σ's lies much more to the right in other subdistricts. Consequently the districts of the government of Tambof have their σ much more on the left side than 58 districts of other governments. Fourthly, σ's for the whole of each government are nearly the same as those of districts notwithstanding that their area is much larger.

The distributions represented in Table I were dissected by the method of the nonic and the results are given below. We see from them that the distribution for East Russians can be resolved in three ways corresponding each to a root of the nonic.

The frequency for East Russians in Table I leads to the nonic

$$p_2^9 + 50{\cdot}675p_2^7 + 9{\cdot}712p_2^6 + 1108{\cdot}501p_2^5 + 396{\cdot}812p_2^4 - 3083{\cdot}708p_2^3 - 5650{\cdot}491p_2^2 - 2428{\cdot}106p_2 - 271{\cdot}467 = 0.$$

The Sturm's functions give:

$f(\infty) = +$	$f(-\infty) = -$
$f_1(\infty) = +$	$f_1(-\infty) = +$
$f_2(\infty) = -$	$f_2(-\infty) = +$
$f_3(\infty) = +$	$f_3(-\infty) = +$
$f_4(\infty) = +$	$f_4(-\infty) = -$
$f_5(\infty) = +$	$f_5(-\infty) = +$
$f_6(\infty) = +$	$f_6(-\infty) = -$
$f_7(\infty) = -$	$f_7(-\infty) = +$
$f_8(\infty) = -$	$f_8(-\infty) = +$
$f_9(\infty) = -$	$f_9(-\infty) = -$
One root between + 2 and + 3.	Two roots between 0 and − 1.

We have

1st solution $p_2 = 2{\cdot}042$,
2nd solution $p_2 = -0{\cdot}179$,
3rd solution $p_2 = -0{\cdot}389$,

* I do not think that it can be explained by the higher variation of the brachycephalic races (see K. Pearson, "Variation in Man and Woman." *The Chances of Death*, Vol. I. p. 292).

giving for: $p_2 = 2{\cdot}042$; $j_1 = 2{\cdot}189$; $j_2 = 0{\cdot}933$; $z_1 = -0{\cdot}7428$; $z_2 = 1{\cdot}7428$;
$v_1 = 7{\cdot}6407$; $\sqrt{v_1} = 2{\cdot}7642$; $v_2 = 9{\cdot}4698$; $\sqrt{v_2} = 3{\cdot}0773$;

for: $p_2 = -0{\cdot}179$; $j_1 = 0{\cdot}9428$; $j_2 = -0{\cdot}1898$; $z_1 = 0{\cdot}1676$; $z_2 = 0{\cdot}8324$;
$v_1 = 12{\cdot}8398$; $\sqrt{v_1} = 3{\cdot}5833$; $v_2 = 7{\cdot}7554$; $\sqrt{v_2} = 2{\cdot}7849$;

for: $p_2 = -0{\cdot}389$; $j_1 = 9{\cdot}4928$; $j_2 = -0{\cdot}0410$; $z_1 = 0{\cdot}0043$; $z_2 = 0{\cdot}9957$;
$v_1 = -0{\cdot}8227$; $\sqrt{v_1} = 0{\cdot}9070$; $v_2 = 8{\cdot}4374$; $\sqrt{v_2} = 2{\cdot}9047$.

(1) If p_2 is 2·042, the curve can be treated as the difference of two curves, the distances of the modal ordinates of which are 2·2 and 0·9.

(2) If $p_2 = -0{\cdot}179$, the compound curve is the sum of two curves the modes of which do not differ much.

(3) If $p_2 = -0{\cdot}389$, the curve is also the sum of two curves but with extremely different modes. The number of cases of the first of the components (extremely brachycephalic) is very low. As we do not find anywhere a type with such a high index (88), I adopted the second solution as correct.

The frequency for the Intermediate Zone is given in Table I and leads to the nonic:

$$p_2^9 - 26{\cdot}827\,p_2^7 + 1{\cdot}103\,p_2^6 + 209{\cdot}959\,p_2^5 - 116{\cdot}425\,p_2^4 - 619{\cdot}249\,p_2^3 - 105{\cdot}775\,p_2^2 + 16{\cdot}574\,p_2 - 0{\cdot}397 = 0.$$

The Sturm's functions are:

$f(\infty) = +$	$f(-\infty) = -$
$f_1(\infty) = +$	$f_1(-\infty) = +$
$f_2(\infty) = +$	$f_2(-\infty) = -$
$f_3(\infty) = +$	$f_3(-\infty) = +$
$f_4(\infty) = +$	$f_4(-\infty) = -$
$f_5(\infty) = -$	$f_5(-\infty) = -$
$f_6(\infty) = +$	$f_6(-\infty) = -$
$f_7(\infty) = +$	$f_7(-\infty) = +$
$f_8(\infty) = +$	$f_8(-\infty) = -$
$f_9(\infty) = -$	$f_9(-\infty) = -$
One positive root between 4 and 5,	Two negative roots between 0 and −2,

and we have

1st solution $p_2 = 4{\cdot}312$,
2nd solution $p_2 = -0{\cdot}294$,
3rd solution $p_2 = -1{\cdot}562$,

giving for: $p_2 = 4{\cdot}312$; $j_1 = 1{\cdot}705 + 1{\cdot}185\,i$; $j_2 = 1{\cdot}705 - 1{\cdot}185\,i$.

This is an imaginary solution so that z_1, z_2, v_1, v_2 ... were not calculated;

for: $p_2 = -0{\cdot}294$; $j_1 = 4{\cdot}289$; $j_2 = -0{\cdot}069$; $z_1 = 0{\cdot}016$; $z_2 = 0{\cdot}984$;
$v_1 = 0{\cdot}369$; $v_2 = 10{\cdot}712$; $\sqrt{v_1} = 0{\cdot}607$; $\sqrt{v_2} = 3{\cdot}273$;

for: $p_2 = -1{\cdot}562$; $j_1 = 1{\cdot}999$; $j_2 = -0{\cdot}781$; $z_1 = 0{\cdot}281$; $z_2 = 0{\cdot}719$;
$v_1 = 8{\cdot}103$; $v_2 = 9{\cdot}740$; $\sqrt{v_1} = 2{\cdot}847$; $\sqrt{v_2} = 3{\cdot}121$.

The distribution of the Intermediate Zone (with the number of cases 1133) affords only two solutions. The first of them gives σ_1 and σ_2 imaginary ($p_2 = 4{\cdot}312$), so that the distribution cannot be treated as a difference. As a sum it gives two solutions:

(1) If $p_2 = 0{\cdot}294$, it is the sum of two components, of which the first has a very high index (85), a very low number of cases and an exceptionally low variation (0·607), and the second has the index 81.

(2) If $p_2 = -1{\cdot}562$, the distribution is the sum of a component with the index nearly 84 (70 °/₀ of the population) and another one with the index 81 (30 °/₀).

Lastly the curve of the east part of the Waldaï Region (number of cases 1076) affords also three solutions.

The frequency for the East part of the Waldaï Region is given in Table I and leads to the nonic:

$$p_2^9 + 69{\cdot}639p_2^7 + 8{\cdot}835p_2^6 + 2157{\cdot}846p_2^5 - 3716{\cdot}901p_2^4 - 10814{\cdot}611p_2^3 - 10086{\cdot}293p_2^2 - 2761{\cdot}091p_2 - 204{\cdot}337 = 0.$$

The Sturm's functions are

$f(\infty) = +$	$f(-\infty) = -$
$f_1(\infty) = +$	$f_1(-\infty) = +$
$f_2(\infty) = -$	$f_2(-\infty) = +$
$f_3(\infty) = +$	$f_3(-\infty) = +$
$f_4(\infty) = +$	$f_4(-\infty) = -$
$f_5(\infty) = +$	$f_5(-\infty) = +$
$f_6(\infty) = +$	$f_6(-\infty) = -$
$f_7(\infty) = -$	$f_7(-\infty) = -$
$f_8(\infty) = -$	$f_8(-\infty) = +$
$f_9(\infty) = -$	$f_9(-\infty) = -$

One root between 2 and 3. Two roots between 0 and − 1.

These lead to:

1st solution $p_2 = 2{\cdot}980$,
2nd solution $p_2 = -0{\cdot}117$,
3rd solution $p_2 = -0{\cdot}270$,

giving for: $p_2 = 2{\cdot}980$; $j_1 = 2{\cdot}603$; $j_2 = 1{\cdot}145$; $z_1 = -0{\cdot}785$; $z_2 = 1{\cdot}785$;
$v_1 = 9{\cdot}194$; $\sqrt{v_1} = 3{\cdot}032$; $v_2 = 11{\cdot}412$; $\sqrt{v_2} = 3{\cdot}378$;

for: $p_2 = -0{\cdot}117$; $j_1 = 0{\cdot}855$; $j_2 = -0{\cdot}137$; $z_1 = 0{\cdot}138$; $z_2 = 0{\cdot}862$;
$v_1 = 15{\cdot}758$; $\sqrt{v_1} = 3{\cdot}970$; $v_2 = 9{\cdot}144$; $\sqrt{v_2} = 3{\cdot}024$;

for: $p_2 = -0{\cdot}270$; $j_1 = 14{\cdot}529$; $j_2 = -0{\cdot}019$; $z_1 = 0{\cdot}001$; $z_2 = 0{\cdot}999$;
$v_1 = -17{\cdot}794$; $\sqrt{v_1} = 4{\cdot}218\,i$; $v_2 = 9{\cdot}940$; $\sqrt{v_2} = 3{\cdot}153$,

an imaginary solution.

(1) If $p_2 = 2{\cdot}980$ it is the difference of two components, one of which has the mode on 85 and the other on 87. Such a brachycephalic type as the latter is nowhere to be found.

(2) If $p_2 = -0{\cdot}117$, the most probable solution is that the distribution is composed of two components: one with the mode on 84 and the second with the mode on 85. The last is appropriate to the population of the central parts of the Waldaï Region.

(3) If $p_2 = -0{\cdot}270$, one of the components shows an imaginary variation.

Those of the above-mentioned solutions which I suppose to be true for each of the three cases respectively are shown in Table I where are also placed for each component: (1) the number of observations (letter n, percents. in brackets); (2) the standard deviation (σ); (3) the mean index (letter m); (4) the criterion (κ). We deduce from this table the following results:

(1) The criterion attains its minimum in the region of the greatest mixture and increases regularly from here to the East and West, i.e. towards the centres of both extreme regions.

(2) The first moment is positive in the three Eastern zones and negative in the two Western. It shows I think that the skewness in the Eastern zones arises from the mixture of an aboriginal dolichocephalic population with a certain number of brachycephalic invaders, and in the Western it is the contrary.

(3) The dolichocephalic component of the Riazan (Eastern) region has nearly the same index and standard deviation as the above-mentioned native Finnish population—Mordwa.

The general deduction from these results is I think that the cephalic index, when inherited, shows intermediate degrees and no mendelisation. This property explains the minimum value of κ in the middle zone where both types were the longest time in contact*.

Table XIII shows us κ and other constants for the different curves represented in Table XII. One part of these curves was formed artificially by addition of empirical curves (the numbers of which are indicated in brackets). We see that an artificial mixture of Nos. 7 and 2 (brachycephalic Kostroma region and rel. dolichocephalic East region) gives κ higher than in each of the components, but the mixture of extreme cases: dolichocephalic Mordwa and brachycephalic Centre of the Waldaï Region, shows κ lower than in each component, so that it is a criterion of the form, and not of the compoundness of a curve. The distribution 10 was formed by addition of 16 distributions for the governments of Great Russia (in percents.). This compound distribution has κ practically as low as a normal curve. Its β is

* The following results were obtained by myself from the observations on families:

TABLE XI.

Index...	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	Totals
Parents	1	3	1	1	2	1	1	—	—	3	5	—	—	1	1	—	20
Children	—	—	2	—	1	4	1	4	4	3	2	1	—	—	—	2	24
Parents	—	—	—	—	—	5	5	—	—	—	—	—	—	—	—	—	10
Children	—	2	1	1	4	1	3	2	—	2	—	—	—	—	—	—	16

TABLE XII. *The Distributions of Cephalic Index, the Constants of which are given in Table XIII.*

	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
(7)* The Government of Kostroma	—	—	—	—	—	1	—	—	4	3	6	12	28	51	71	142	205	260
(8) The Government of Kostroma and the East Region†	—	—	—	—	—	1	2	5	13	32	69	144	281	386	618	781	1078	1119
(9) Mordwa-moksha Tribe and the Centre of the Waldaï Region† ...	—	—	—	—	—	—	—	—	—	51	17	86	241	454	622	1061	1057	810
(10) Sixteen Governments of Great Russia†	—	—	—	—	—	1	3	4	10	14	43	88	184	244	461	656	957	1109
(11) The East Region and the Centre of the Waldaï Region†	—	—	—	—	—	—	2	5	7	27	59	125	236	378	503	663	1044	922
(12) The West of the East Region and the East of the Waldaï Region†	—	—	—	—	—	2	—	6	8	6	41	92	176	222	421	560	982	1068
(13) The Governments of Toula and Moscow†	—	—	—	—	—	—	—	—	—	35	79	53	159	309	459	574	962	1059
(14) The whole of the Intermediate Zone†	—	—	—	—	—	1	—	4	6	16	54	79	170	251	433	565	975	1065
(15) The Government of Smolensk	—	—	—	—	—	—	1	—	3	2	11	12	33	31	82	124	225	320
(16) The District of Spassk of the Government of Tambof (Russians)	—	—	—	—	—	—	—	—	2	—	2	7	20	21	23	41	42	33
(17) The Barrow-skulls of Great Russia	—	—	—	2	3	8	4	15	23	34	31	21	22	28	22	13	16	15
(18) The same in the South of Kostroma	—	1	—	—	—	2	4	2	4	9	13	6	9	10	11	16	10	8
(19) The same in Little Russia	—	—	1	—	—	1	1	6	7	15	14	20	20	18	13	11	6	6
(20) The East Region artificially made symmetrical	—	—	—	—	—	—	3	5	14	19	36	95	188	365	461	655	875	1091

	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	Totals
(7)* The Government of Kostroma	295	349	396	355	287	219	143	117	76	28	23	7	4	2	—	3084
(8) The Government of Kostroma and the East Region†	1038	985	937	809	586	416	255	201	132	49	39	14	6	3	—	10000
(9) Mordwa-moksha Tribe and the Centre of the Waldaï Region† ...	894	756	913	663	927	408	390	353	186	37	74	—	—	—	—	10000
(10) Sixteen Governments of Great Russia†	1129	1117	1105	936	713	472	303	203	132	52	36	10	9	4	—	9995
(11) The East Region and the Centre of the Waldaï Region†	970	830	967	793	978	434	396	348	196	41	76	—	—	—	—	10000
(12) The West of the East Region and the East of the Waldaï Region†	1249	985	1038	1061	708	520	354	261	105	61	36	2	22	9	—	9995
(13) The Governments of Toula and Moscow†	1233	1085	1174	794	777	424	344	194	159	—	18	—	9	—	—	10000
(14) The whole of the Intermediate Zone†	1277	1019	1083	972	731	488	354	239	123	40	30	1	18	6	—	10000
(15) The Government of Smolensk	364	306	302	285	224	163	89	85	42	28	8	—	6	1	—	2747
(16) The District of Spassk of the Government of Tambof (Russians)	33	19	16	6	3	3	3	1	—	—	1	—	—	—	—	276
(17) The Barrow-skulls of Great Russia	8	11	7	2	1	—	—	1	—	—	—	—	—	—	—	288
(18) The same in the South of Kostroma	5	6	3	4	4	2	1	—	—	—	—	—	—	—	—	129
(19) The same in Little Russia	4	5	5	4	—	—	—	—	1	—	—	—	—	—	—	158
(20) The East Region artificially made symmetrical	875	655	461	365	188	95	36	19	14	5	3	—	—	—	—	6523

* For these numbers see Table XIII. † Percents. multiplied by 100.

TABLE XIII.

The Constants of Distributions in Table XII.

	μ_1	μ_3	μ_2	μ_4	β_1	β_2	κ	N	Mode
1. Mordwa-moksha Tribe	0·965517	6·001824	6·950536	165·259577	0·111231	3·431270	0·162407	290	78
2. The Eastern Region	0·4636	2·5446	8·8698	250·4923	0·0095	3·1875	0·020628	7815	79
3. Its Western Zone	1·554994	2·481443	9·984017	332·370884	0·006345	3·340521	0·007219	2373	79
4. The Intermediate Zone	0·565755	−0·857468	10·925287	350·370763	0·000577	2·934793	0·003276	1133	81
5. The East of the Waldaï Region	−0·943308	2·426933	10·257839	335·7197	0·005592	3·192204	0·011434	1076	84
6. The Centre of the Waldaï Region	−1·126866	−3·792394	9·304800	237·631898	0·018341	2·740135	−0·024103	134	85
7. The Government of Kostroma	0·2646	−2·1464	11·5247	420·9952	0·00308	3·17222	0·006902	3084	83
8. The Government of Kostroma and the East Region (2+7)...	1·1030	6·963374	12·868391	489·417654	0·022874	2·955225	0·109101	10899	79
9. Mordwa-moksha and the Centre of the Waldaï Region (1+6)	3·4183	9·786419	14·129325	481·630571	0·034561	2·405572	0·020521	424	78
10. Great Russia (16 Governments)	0·630304	2·333719	11·48094	411·796576	0·003678	3·126018	0·011461	—	81
11. The East Region and the Centre of the Waldaï Region (2+6)	2·6678	0·773158	13·948643	504·202552	0·002234	2·586557	0·002022	7949	79
12. The West of the Eastern and the East of the Waldaï Regions	0·8049	2·921775	11·658036	427·301894	0·005505	3·146154	0·014999	3449	81
13. The Governments of Toula and Moscow (3+5)	0·5667	−0·786984	10·926351	350·830186	0·000485	2·930823	0·001928	1133	81
14. The whole of the Intermediate Zone (3+4+5)	0·7248	1·773883	11·428465	404·636345	0·002155	3·099889	0·008367	4582	81
15. The Government of Smolensk	1·1041	5·4905	10·9415	378·5546	0·02355	3·16464	0·06872	2747	81
16. The Spassk District (Russians)*	0·148550	10·034181	8·785904	293·638425	0·152764	3·819576	0·101491	276	79
17. The Barrow-skulls of Great Russia	2·9549	19·5190	16·4804	666·8860	0·08642	2·44983	−0·04947	288	72
18. The same in the South of Kostroma	−1·1163	11·4614	18·3818	738·2085	0·02144	2·17733	−0·00972	129	78
19. The same in Little Russia	0·6709	28·6088	14·1195	704·0689	0·29597	3·53800	1·267752	158	75
20. The East Region artificially made symmetrical	0·0000	0·0000	8·1586	176·8630	0·0000	2·650739	0·0000	6523	80
Mean values:									
A. 21. 5 mixed curves (8, 10, 12, 13, 14)	0·7659	—	11·672	416·7965	(408·7068)	3·0517	0·0292	—	—
B. 22. 6 simple curves (2, 3, 4, 5, 7, 15)	0·8160	—	10·4171	344·7505	(325·5480)	3·1653	0·0197	—	—

* The District of Spassk of the Tambof Government.

also low, but the difference between μ_4 and $3\mu_2^2$ is 16·3633. It would be not without interest to resolve this distribution by the aid of nonic.

The same Table XIII shows us the mean constants of 6 empirical curves and of 5 others which were artificially formed from them by addition (the numbers of components are indicated in brackets). We see that mean κ's in both groups tend towards zero and β_2's towards 3, as in a normal curve. The mean fourth moment of artificially made compound curves is nearer to $3\sigma^2$ than that of their components. The first moments are also not different. But the second moment is higher in compound curves. Other mean constants tend to the values which characterize the normal curve, as a state of equilibrium.

As a general conclusion I wish to call attention to the importance for anthropology of the standard deviation and of the methods to decide with certainty whether a distribution is compound or not. The determination of its components ought to be made without using the method of correlation, as there may be no correlation between characters in single individuals, as we have seen above.

Twenty-five years ago I was happy in being the first to call the attention of Russian naturalists to the methods of Biometry (in the *Proceedings* of the Russian Anthropological Society). Since then more competent authors have printed manuals, criticisms or mathematical modifications* of the methods and applied them to many investigations in different branches of Russian science. In Anthropology these methods have been applied mostly by myself and my pupils—students of different Universities—during practical training in anthropometry and afterwards

TABLE XIV.

Permiaks. Correlation between Standard Deviations of Index and of Stature (Means of Subdistricts). After Mr Wishnewsky.

	33	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	Totals
29	—	—	—	—	—	—	—	—	—	—	1	—	1
30	—	—	—	—	—	—	—	—	—	—	—	—	—
31	—	—	—	—	—	—	—	—	—	—	—	1	1
32	—	—	—	—	—	—	—	—	—	1	—	—	1
33	—	—	—	—	—	—	—	—	—	—	—	—	—
34	—	—	—	—	—	1	—	—	—	—	—	—	1
35	—	—	—	1	—	—	—	—	—	—	—	—	1
36	—	—	—	—	—	—	—	—	—	—	—	—	—
37	—	—	—	1	—	—	—	—	—	—	—	—	1
38	1	—	1	—	—	—	—	—	—	—	—	—	2
39	—	—	—	—	—	—	—	—	—	—	—	—	—
40	—	—	—	—	—	—	—	—	—	—	—	—	—
41	—	—	—	—	—	—	1	—	—	—	—	—	1
42	—	—	—	—	—	—	—	—	—	—	—	—	—
43	—	1	—	—	—	—	—	—	—	—	—	—	1
Totals	1	1	1	2	—	1	1	—	—	1	1	1	10

* Many mathematical studies concerning coefficients of correlation and dispersion etc. have appeared in Russian statistical papers during the last three years. I hope to give some account of them later.

in special papers. Among others, Boris Wishnewsky (formerly lecturer in Kazan University, now Curator of the Anthropological Museum of the Academy of Science) has used biometric methods in his description of Permiaks. One of the facts found by him we could not explain: it is a very high negative correlation* between the standard deviations of stature and of cephalic index in subdistricts (Table XIV).

Mr Siedykh has calculated the coefficients of correlation between different characters for couples of sisters. The same individual sister was measured and observed twice a year during nearly 20 years (in a Crown Boarding School for girls) and the date of first menstruum (menarche) was noticed exactly. The marks for different branches of studies were also very carefully determined in the form of means for three last years (from six marks: three mean marks for each year and three for the corresponding examinations). The results which have not hitherto been published are given in Table XV.

TABLE XV.

Correlation between Characters of Sisters.

Character	Means		Standard Deviations		Coefficient of Correlation	Number of Couples
	a	*b*	*a*	*b*		
1. Stature (both sisters at the age of 10)	130·71 cm.	131·59	5·56	4·89	0·41	84
2 Id. Weight	27·21 kg.	27·46	3·01	2·72	0·43	84
3. Stature of the same couples of sisters at the term of menarche	151·01	151·11	5·66	5·87	0·38	84
4. Id. Weight	45·05	44·17	5·40	4·80	0·22	84
5. Stature of the same couples of sisters at the age of 18·19	158·19	158·37	5·07	4·09	0·40	84
6. Id. Weight	54·02	54·29	5·24	4·56	0·32	84
7. Age at menarche	13·51	13·34	1·16	1·17	0·30	282
8. General mean mark† of all branches of studies	10·77	10·94	1·34	1·22	0·36	186
9. Mean mark for Drawing	11·056	11·033	1·00	0·91	0·16	142
10. Id. for Music (piano)	2·717	10·124	1·65	1·63	0·29	113
11. Id. for Mathematics	0·18	10·50	2·77	2·45	0·40	187
12. Id. for French	9·522	9·76	2·28	2·09	0·35	187
13. Id. for German	9·29	9·83	2·39	2·33	0·45	187

b—Younger sister.
a—Older sister.

From the printed work of another student (Doudenkof) I extract the following table representing the correlation between statures of the same girls at the ages of 8 and 19 (Table XVI). The correlation is $+ \cdot 687 \pm \cdot 033$.

* [The correlation is $- \cdot 414 \pm \cdot 279$, i.e. about 2·3 times its probable error. It may be significant, but it requires confirmation on a much larger number of groups. Ed.]

† A 12-mark-system was used. The highest mark is 12, the lowest 1. No. 8 was deduced as a mean from six mean marks obtained during the last three years of studies in the following way: for each year the mean mark was deduced and then a mean obtained from it and the examination mark of the same year. From three such marks the general mean mark (No. 8) was deduced. There was no examination mark for drawing.

TABLE XVI.

Correlation between Statures of the same Girl at the Ages of 8 *and* 19.

Stature at Age 19.

Stature at Age 8.	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	n_1
109	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
110	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
111	1	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
112	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
113	1	—	—	—	—	—	1	2	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
114	—	—	—	—	1	—	—	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	3
115	—	—	—	—	—	1	2	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
116	—	—	—	—	—	—	—	—	—	—	1	1	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	5
117	—	—	—	—	—	—	—	—	1	—	—	2	2	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
118	—	—	—	—	—	—	—	1	1	—	1	2	3	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	11
119	—	—	—	—	—	—	—	—	—	—	2	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	5
120	—	—	—	—	—	—	—	—	1	1	2	—	—	—	—	1	—	—	1	—	—	1	—	—	—	—	—	—	7
121	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2	2	—	—	1	—	—	—	—	—	—	—	—	—	7
122	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	1	1	3	1	—	—	—	—	—	—	—	—	—	11
123	—	—	—	—	—	—	—	—	—	—	—	—	3	—	2	3	1	1	3	—	1	—	—	—	—	—	—	—	14
124	—	—	—	—	—	—	—	—	—	—	1	—	—	—	2	—	—	—	—	1	1	—	—	—	—	—	—	—	5
125	—	—	—	—	—	—	—	—	—	—	—	1	—	2	—	2	2	1	—	—	1	—	—	—	—	—	—	—	9
126	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	2	—	1	—	—	—	—	—	1	—	—	6
127	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	2	—	—	—	—	—	—	—	—	—	—	3
128	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
129	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
130	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	—	—	1	—	—	—	—	—	4
131	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
132	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
n_2	2	—	—	—	1	1	3	3	4	2	10	8	12	13	13	12	7	11	7	2	3	1	1	—	—	1	—	1	118

To this table I add Table XVII with my own observations of the like nature for the cephalic index of the same children measured at the ages of 11 and of 16. The correlation is $+ \cdot 839 \pm \cdot 030$. We perceive from these tables how the individuals having the same degree of character at a given age vary with their growth. I limit myself to these examples concerning biometrical methods in Anthropology in Russia, hoping to give at a later time a more detailed account of what has been attained in biometry by Russian scientific workers.

TABLE XVII.

Correlation between Cephalic Index of the same Child at the Age of 11 *and of* 16.

Cephalic Index at Age 11.

Cephalic Index at Age 16.	*77*	*78*	*79*	*80*	*81*	*82*	*83*	*84*	*85*	*86*	*87*	*88*	*89*	*90*	Totals
75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
77	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
78	—	—	2	—	—	—	—	—	—	—	—	—	—	—	2
79	—	1	2	—	—	—	—	—	—	—	—	—	—	—	3
80	—	—	—	2	2	—	—	—	—	—	—	—	—	—	4
81	—	—	1	—	2	3	3	1	—	—	—	—	—	—	10
82	—	—	—	—	—	1	3	1	1	—	—	—	—	—	6
83	—	—	—	—	—	—	1	2	1	—	—	—	—	—	4
84	—	—	—	—	—	—	—	1	2	1	—	—	—	—	4
85	—	—	—	—	—	—	—	1	2	2	—	1	—	—	6
86	—	—	—	—	—	—	—	—	1	—	1	1	—	—	3
87	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
88	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
89	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
90	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
Totals	1	1	5	2	4	4	7	6	8	4	1	2	—	1	46

Appendix. Composite Photographs.

I have constructed a special photographic apparatus which enables me to take rapidly nearly 2000 portraits on a cinematographic film without recharging. With it I have photographed under the same conditions of light nearly 5000 men 26—27 years old in full face and profile. Using Galton's method I made with a reflex-camera the combined portraits of the above-mentioned East Region population, i.e. of the Riazan, Tambof and Penza Governments (see Plate II, Fig. 1), of White Russians of Mariupol District (Fig. 2) and of Little Russians of the Government of Minsk (Fig. 3). We see from these portraits combined from a large number of individuals (451, 147 and 140 individuals) that only a slight long-headedness characterizes the Eastern type. As a result I found no sensible differences between these types using Bertillon's scheme. The composite portrait of Mongols, combining only 25 individuals, shows a striking difference (Fig. 4). I have also used my apparatus to make composite portraits of the profiles of the ears (Figs. 5*a*, 5*b*) and of the nose (Fig. 6) which as far as I know have not previously been made by anybody else. The events of the last few years have interrupted my investigations.

ON TESTING VARIETIES OF CEREALS.

By "STUDENT."

Being a Paper read before the Society of Biometricians and Mathematical Statisticians. May 28th, 1923.

Object of Experiments.

The object of testing varieties of cereals is to find out which will pay the farmer best. This may depend on quality, but in general it is an increase of yield which is profitable, and since yield is very variable from year to year and from farm to farm it is a difficult matter upon which to obtain conclusive evidence.

Yet it is certain that very considerable improvements in yield have been made as the result of replacing the native cereals by improved varieties; as an example of this I may cite the case of Ireland, where varieties of barley have been introduced which were shown by experiment to have an average yield of 15 % to 20 % above those which they replaced. This represents, probably, a gain to the country of not less than £250,000 per year. As the cost of experiments from the commencement to the present time cannot have reached £40,000 the money has been well spent.

Origin of Varieties.

In the first place the ordinary cereals, wheat, barley, oats, and so on (maize is not here considered), are all self-fertilized and occur in races broadly distinguished by different botanical characters—Potato Oats, Rivett Wheat, Chevalier Barley, and so forth.

Besides these botanically distinguishable races, it is possible to pick out strains from commercial seed which differ from one another in all kinds of ways: time of ripening, percentage of nitrogen, yield, etc., although botanically the same. Many of these strains have been selected from time to time, certainly from the end of the eighteenth century up to the present time.

Finally there are hybrids, the result of deliberate crossing, and the selection of the best individuals out of the many thousands which may be grown in three generations is one of the more difficult problems with which the plant breeder has to deal, but it is only after he has made his preliminary selection that his hybrids concern the experimenter who is testing varieties.

Owing to the fact of self-fertilization, the various races, strains, and even to a large extent the hybrids, remain practically constant from year to year if once pure seed has been obtained.

Chief Sources of Error.

The peculiar difficulties of the problem lie in the fact that the soil in which the experiments are to be carried out is nowhere really uniform; however little it may vary to the eye, it is found to vary not only from acre to acre but from yard to yard, and even from inch to inch. This variation is anything but random, so that the ordinary formulae for combining errors of observation which are based on randomness are even less applicable than usual.

Next, of course, is the weather: that will hardly affect experiments carried out in the same field in the same year, but experiments carried out in different districts and seasons meet with variations of weather which may produce results quite inconsistent with the experimental error determined at either place. Obviously, the weather needs to be well sampled before drawing general conclusions.

The effects of soil and weather on the yields are far greater than the differences which we have to investigate, and it is because the planning of experiments and their interpretation when completed are not quite straightforward that this paper has been written.

Methods of Operating.

There are, broadly speaking, two methods of operating:

i. On a large enough scale to use the ordinary agricultural implements, ploughs, seed drills, reaping machines, etc.

ii. On quite a small scale with spades and dibblers, and scissors, under a wire net to keep out birds and rabbits.

Taking first the large scale, it has the advantage that the farmer, who always has a healthy contempt for gardening, may pay some attention to the results; he is to this extent right, that large scale conditions cannot be accurately reproduced in a wire cage, and in fact some varieties which have come out well on the small scale have not done as well in the field, though this is not at all common. Large-scale work then, is necessary as a final demonstration, and historically, it was on the large scale that variety experiments were first carried out.

Large Scale Work.

As an instance of large scale work, we may take a series of experiments carried out by the Department of Agriculture in Ireland to find out the best variety to grow in that country.

The experiments lasted six years, *vide* Table I, and during that time seven varieties were tested; only two, however, Archer and Goldthorpe, were carried through from start to finish, as the others were either dropped when they were found to be inferior, or were not among those chosen in the first place. The original seed was ordinary commercial seed, and the plots were two acres in extent. This is very large even for a large scale plot, but it was intended that the produce should form the raw material for further manufacturing experiments. This was a wise precaution, as has been found recently when a barley in other ways among the best was found to be quite unsuitable as malting material.

The produce of the plots was all valued (in those days—1901–1906—values were fairly steady from year to year), and this gives a method of combining yield and quality, but although the quality varied very much from one farm to another, there was generally only a small difference between the quality of different varieties grown on the same farm in the same season. The value of the crop per acre depended chiefly on the yield.

During the six years 193 plots were grown and at different times eighteen farms provided the land. These farms were scattered up and down the barley-growing districts in Ireland. Here, however, we shall deal only with the 51 plots of Archer, and the corresponding 51 plots of Goldthorpe.

The value per acre, then, of the 51 Archer plots varied between 90/- and 234/- with a mean of 178/- and a standard deviation of 33·6 shillings. The value per acre of the Goldthorpe plots varied between 99/- and 230/- with a mean of 166/- and a standard deviation of 33/-. The difference, therefore, was 12/-, and at first sight this hardly appears significant, for had the Archer and Goldthorpe plots been independent, the standard deviation of their difference would have been about 6·5.

This brings us to the first principle of all agricultural experiments, viz., that only comparative values are of any use. If we are told that on a certain farm a new variety of barley produced 30 cwts. to the acre, we admit that the crop is good, but are not much interested. If, in addition, we hear that Archer gave 25 cwts. to the acre on the same farm, we begin to take notice; for it is some evidence as to the value of the new variety, and it is the difference of 5 cwts. to the acre which appeals to us and not the actual yields themselves. In point of fact, of course, the yields in these experiments were not independent. Each Archer has a corresponding Goldthorpe, and by considering the 51 differences, we find that the mean difference between Archer and Goldthorpe has a standard deviation of 3·3 shillings.

This reduction of the S.D. of the mean difference from 6·5 to 3·3 shillings, by considering the individual differences between corresponding pairs, depends of course on the fact that corresponding pairs are highly correlated, so that the last term in the formula

$$\sigma^2_{A-B} = \sigma^2_A + \sigma^2_B - 2r_{AB}\sigma_A\sigma_B$$

is by no means negligible. The art of designing all experiments lies even more in arranging matters so that r_{AB} is as large as possible, than in reducing σ^2_A and σ^2_B.

That the conclusion that Archer was better than Goldthorpe was fully justified is shown by the fact that taking the yearly averages Archer beat Goldthorpe every year, while in the individual farms Archer beat Goldthorpe in all but three out of eighteen, and of these one farm was only used one season, and the other two in two seasons. Further, it was discovered during the course of the experiments that the Archer was practically identical with a barley which the Danes called Prentice, which had beaten all others in their long series of experiments. Both Archer and Goldthorpe were, practically speaking, new to Ireland, and they—or some improve-

ment* on them—have now almost entirely driven out the other inferior barleys from most parts of the country.

Such, then, is the sort of error which attaches to large experimental plots, that is to say a standard deviation of about 10–15% for a single comparison, and this is found to be the order of the error in all ordinary large scale work—it does not vary very closely with the size of the plot, provided that the plot be above say one-tenth of an acre, though there may be a slight decrease of error with increase of size.

It follows that although it is quite within the power of any individual farmer to carry out a large scale experiment (and the larger the easier to carry out), it is only by co-operation that enough evidence can be obtained to be of any value. This co-operation can in practice only be arranged by a government department, a large agricultural company, or a farmers' association, and it is government departments that have had most success.

Small Scale Work.

We may next discuss small scale work, leaving to the end a modification introduced by Dr E. S. Beaven, which combines the advantages of the ordinary large scale with a considerably smaller error. The considerations which led to this modification were derived from experience of small scale technique.

Preliminary Considerations. Before coming to any actual comparison of varieties on the small scale, attention is directed to some preliminary experiments carried out by three different sets of investigators: Stratton and Wood† at Cambridge, Mercer and Hall at Rothamsted‡, and Montgomery at Nebraska Agricultural Experimental Station§.

The first harvested 9/10th acre of mangolds in 1/1000-acre plots: the second, one acre of wheat in 1/500-acre plots, and an acre of mangolds in 1/200-acre plots: the third two years in succession harvested the same 7/45th acre of wheat in 1/1440-acre plots, and all weighed the produce of each plot; Montgomery determined the percentages of nitrogen as well. All three experiments showed the same thing: that the variation is not random; the yield varies from point to point with an irregular regularity; there is consequently correlation between one plot and its neighbours, and generally there is a tendency for one end of a field to yield more than the other.

This is only what is to be expected from *a priori* considerations; naturally the nearer two plots are together the more likely is the soil and its condition to be

* In particular a hybrid of Archer with Spratt made by Capt. Hunter, Spratt-Archer 37/6, which proved its superiority to Archer and other varieties in "chessboard" trials similar to that detailed below.

† *Journal of Agricultural Science*, Vol. III. p. 417, "The Interpretation of Experimental Results."

‡ *Journal of Agricultural Science*, Vol. IV. p. 107, "The Experimental Error of Field Trials."

§ *Nebr. Agr. Exp. Sta. 25th Ann. Report*, 1910—11, pp. 164—180, "Variation in Yield and Methods of arranging Plots to secure comparative Results"; and *U.S. Dept. Agr. Bur. Plant. Indus. Bul.* 269, "Experiments in Wheat Breeding: Experimental Error in the Nursery and Variation in Nitrogen and Yield."

similar on each of them, and the obvious conclusion may be drawn that the smaller the plots the more exactly can the yield of adjacent plots be compared.

Taking the investigation of Mercer and Hall on the 500 "plots" of wheat, it should be noted that they were only taken as plots at harvest and before cutting formed an unusually uniform area of one acre, part of a much larger field of wheat. The mean yield of grain per plot was 3·95 lbs. with a range of 2·75—5·14, and a standard deviation of ·46 lb., or 11·6% of the mean weight of a plot.

If two adjacent plots were taken as $\frac{1}{250}$ ac. plots the S.D. fell to 10% instead of the 8·2% of random sampling.

If four adjacent plots were taken as $\frac{1}{125}$ ac. plots the S.D. fell to 8·9% instead of the 5·8% of random sampling.

If ten adjacent plots were taken as $\frac{1}{50}$ ac. plots the S.D. fell to *6·3% instead of the 3·7% of random sampling.

If twenty adjacent plots were taken as $\frac{1}{25}$ ac. plots the S.D. fell to *5·7% instead of the 2·6% of random sampling.

If fifty adjacent plots were taken as $\frac{1}{10}$ ac. plots the S.D. fell to *5·1% instead of the 1·6% of random sampling.

The high value of the S.D. of the larger plots compared with that which would have been expected had the aggregation been carried out randomly is due to a similar cause to that which decreased the error of the comparison of Archer and Goldthorpe. There is correlation between the neighbouring small plots which make up the larger plots, so that the last term in the formula

$$\sigma^2_{A+B} = \sigma^2_A + \sigma^2_B + 2r_{AB}\sigma_A\sigma_B$$

is not negligible. This last term is in fact the bridge over a pitfall which has trapped many, including—as will be shown later—the present writer.

In an appendix to Mercer and Hall's paper I pointed out that advantage may be taken of this correlation if we consider the difference between adjacent plots.

Thus we have:

Size of plot	S.D. of single plot as per cent.	Calculated S.D. of difference between random pairs	Actual S.D. of difference between adjacent pairs	Total acreage required to reduce S.D. of a comparison to 1 per cent.
1/500	11·6	16·4	11·2	·50 acre
1/250	10·0	14·1	9·7	·74 "
1/125	8·9	12·6	9·3*	1·37 "
1/50	6·3	8·9	3·7*	1·10 "
1/25	5·7	8·1	3·9*	3·84 "

Except in the case of the 1/125 plots we actually find that the standard deviation of a difference between two plots is less than the standard deviation of a single plot, and that working with 1/500 plots, the standard deviation of a

* The numbers are too few to do much more than indicate the tendency.

TABLE I.

Irish Experimental Barley Plots. Yield and Money Value Per Acre of Archer and Goldthorpe 1901—1906.

Farmer	Place	District	Archer					Goldthorpe				
			Yield		V. P. A.			Yield		V. P. A.		
			Barrels	Stones	£	s.	d.	Barrels	Stones	£	s.	d.
1901 :												
McCarthy ...	Ballinacurra .	Cork	11	4	9	0	0	7	0	5	2	0
Hawkins ...	Whitegate ...	„	10	3	8	3	0	7	12	6	3	0
Dwan ...	Thurles ...	Central Plain	15	2	11	13	0	13	14	11	0	0
Wolfe ...	Nenagh ...	„	11	0	8	13	0	10	0	8	3	0
1902 :												
McCarthy ...	Ballinacurra .	Cork	12	6	8	13	0	11	14	8	11	0
Hawkins ...	Whitegate ...	„	14	0	10	12	0	13	0	10	3	0
Wolfe ...	Nenagh ...	Central Plain	12	2	9	4	0	13	6	10	2	0
Willington...	Birr... ...	„	12	6	9	16	0	9	3	7	6	0
Gorman ...	Enniscorthy .	Wexford	11	5	9	2	0	11	14	9	2	0
Nunn ...	Castlebridge .	„	11	3	8	18	0	11	4	9	0	0
1903 :												
McCarthy ...	Ballinacurra .	Cork	6	10	4	13	0	7	4	5	5	0
Hawkins ...	Whitegate ...	„	8	12	7	1	0	7	5	5	19	0
Wolfe ...	Nenagh ...	Central Plain	8	2	5	9	0	8	7	6	11	0
Willington...	Birr... ...	„	9	13	7	9	0	8	0	6	6	0
Gorman ...	Arnestown ...	Wexford	5	5	4	10	0	7	11	5	15	0
Nunn ...	Castlebridge .	„	12	7	9	16	0	9	15	7	16	0
Quinn ...	Carlingford...	Louth	11	12	8	4	0	9	3	7	0	0
Kearney ...	Greenore ...	„	11	3	8	12	0	7	13	5	19	0
1904 :												
McCarthy ...	Ballinacurra .	Cork	10	4	7	15	0	11	14	9	8	0
Hawkins ...	Whitegate ...	„	10	11	8	7	0	10	4	8	4	0
Wolfe ...	Nenagh ...	Central Plain	13	3	10	9	0	11	8	9	7	0
Willington...	Birr... ...	„	11	3	8	17	0	11	14	9	4	0
Kelly ...	Portarlington	„	12	1	9	12	0	11	3	9	0	0
Allardyce ...	Monasterevan	„	10	7	8	1	0	10	5	8	7	0
Roche ...	New Ross ...	Wexford	8	2	5	16	0	7	0	5	6	0
Nunn ...	Castlebridge .	„	9	2	7	8	0	6	4	4	19	0
Kearney ...	Carlingford...	Louth	8	0	6	7	0	9	7	7	9	0
Segrave ...	Dunleer ...	„	12	1	9	9	0	11	7	9	7	0
1905 :												
McCarthy ...	Ballinacurra .	Cork	12	8	9	8	0	13	1	9	16	0
Hawkins ...	Whitegate ...	„	11	11	9	8	0	11	5	8	14	0
Wolfe ...	Nenagh ...	Central Plain	14	6	10	14	0	15	10	10	3	0
Willington...	Birr... ...	„	14	11	11	14	0	13	8	10	11	0
Luttrell ...	Monasterevan	„	14	8	11	0	0	12	13	9	14	0
Kelly ...	Portarlington	„	12	1	8	19	0	10	8	7	17	0
Matthews ...	Tullamore ...	„	13	12	10	18	0	10	10	8	1	0
Nunn ...	Castlebridge .	Wexford	11	6	7	15	0	11	6	8	0	0
Dooley ...	New Ross ...	„	13	0	10	0	0	13	10	10	12	0
Kearney ...	Carlingford...	Louth	14	6	9	12	0	11	4	7	12	0
Segrave ...	Dunleer ...	„	14	7	11	11	0	12	8	9	19	0
1906 :												
McCarthy ...	Ballinacurra .	Cork	9	14	7	6	0	9	11	7	2	0
Hawkins ...	Whitegate ...	„	10	9	7	12	0	8	14	6	9	0
Wolfe ...	Nenagh ...	Central Plain	11	12	8	14	0	8	13	6	11	0
Willington...	Birr... ...	„	10	15	8	0	0	9	15	7	6	0
Luttrell ...	Monasterevan	„	9	10	7	3	0	10	9	7	17	0
Mulhall ...	„	„	12	8	9	6	0	13	14	10	7	0
Matthews ...	Tullamore ...	„	8	14	6	16	0	8	11	6	11	0
Tennant ...	Bagnalstown.	„	15	4	11	7	0	13	14	10	9	0
Nunn ...	Castlebridge .	Wexford	11	10	8	19	0	10	9	7	18	0
Dooley ...	New Ross ...	„	14	3	10	7	0	12	5	9	0	0
Kearney ...	Carlingford .	Louth	11	11	8	9	0	12	12	9	8	0
Segrave ...	Dunleer ...	„	14	6	10	8	0	13	6	9	16	0

NOTE. The Irish barrel of barley contains 16 stones.

DIAGRAM I.

Giving the Yields in Grammes of the Individual Plots of Dr Beaven's No. 1 *Yield Experiment of* 1913.

Direction described as "vertical" ——→

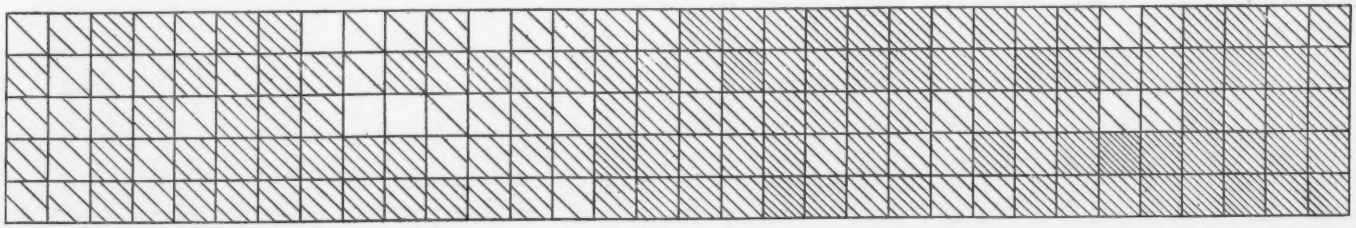
E 230·1	B 249·3	G 312·2	D 263·4	A 263·4	F 273·4	C 314·9	H 198·0	E 227·9	B 225·7	G 265·4	D 263·0	A 236·9	F 228·0	C 293·8	H 237·6	E 336·6	B 330·2	G 323·5	D 341·0	A 327·0	F 308·1	C 324·9	H 298·7	E 334·8	B 316·6	G 258·2	D 299·8	A 338·9	F 321·0	C 317·2	H 266·8
D 255·9	A 222·6	F 218·7	C 258·2	H 273·3	E 268·2	B 301·1	G 285·7	D 212·0	A 296·7	F 237·7	C 308·8	H 225·0	E 290·4	B 297·5	G 334·4	D 282·6	A 374·3	F 309·4	C 377·9	H 303·0	E 351·5	B 344·7	G 317·7	D 362·4	A 331·8	F 298·7	C 366·4	H 325·6	E 377·0	B 354·7	G 321·8
C 265·6	H 205·0	E 246·7	B 277·4	G 232·3	D 286·0	A 286·1	F 229·3	C 221·2	H 170·7	E 231·8	B 235·5	G 264·4	D 239·7	A 302·2	F 278·8	C 391·2	H 260·3	E 328·9	B 324·1	G 305·9	D 308·7	A 248·0	F 272·7	C 324·2	H 295·6	E 253·5	B 286·7	G 372·6	D 322·3	A 362·9	F 326·3
B 265·9	G 236·7	D 295·8	A 238·3	F 265·0	C 320·9	H 272·9	E 298·7	B 307·9	G 301·3	D 244·2	A 263·1	F 257·5	C 318·2	H 331·7	E 324·0	B 301·7	G 326·8	D 325·5	A 280·2	F 315·0	C 305·9	H 269·2	E 376·4	B 314·2	G 376·0	D 428·4	A [illegible]	F 362·9	C 379·9	H 348·6	E 371·3
A 236·5	F 210·4	C 291·1	H 223·9	E 295·6	B 290·5	G 286·1	D 266·2	A 291·6	F 247·3	C 321·4	H 249·9	E 286·5	B 246·6	G 301·0	D 333·0	A 322·0	F 282·4	C 399·3	H 330·8	E 303·9	B 343·3	G 285·2	D 273·5	A 333·4	F *295·9	C 362·6	H 349·8	E 395·3	B 400·6	G 364·9	D 361·8

——Direction described as right——→ and left ←——

A. English Archer. B. 145. C. 145/46. D. Plumage.
E. Early Archer. F. Irish Archer. G. 7 *A*. H. Biffin.

DIAGRAM II.

Plan of Experimental Area, showing Variation in Yield in Different Parts. Each line represents 20 *grammes more yield compared with the mean of the race grown on that plot.*



* Owing to a slip of the draughtsman the vertical rules between these asterisks have been omitted; the omission has no significance.

comparison between the varieties grown on a total area of half an acre is as low as 1°/₀. On the lines of the 2-acre plots more than half a square mile would have been required. Further, there is every indication that smaller plots would be still more economical of ground.

These have been termed preliminary experiments, and so they are for the purpose of this paper; but in point of fact they followed the practical application of the principle which has just been outlined, and a further step in advance had already been made.

Carrying the principle of maximum contiguity, which he had deduced *a priori*, to its extreme logical limit, Beaven had compared two varieties in his cage by sowing alternate rows. He used a pure line of Archer barley, and one of a variety called "Plumage," which is allied to the Goldthorpe of the Irish experiments. He also grew 1/10th acre of each outside the cage and found that whereas the Archer gave slightly the better yield outside the cage, the cage work gave the yield of Plumage some 20°/₀ better than the Archer.

He sent me the figures to look at, and I found that so far from the correlation between the yields of adjacent drills being positive, it was significantly negative.

This was quite unexpected at the time (1905), but the explanation was simple, viz., that when a plant of one variety is grown next to one of another variety it is abnormally situated, and is subject to abnormal competition.

In this case the Plumage was a taller barley and shaded the Archer; probably also, it started growth more quickly underground and so annexed more of the soil than its competitor. Anyhow, it was clear that a comparison of adjacent rows, with the possibility of interference of this kind, was useless.

The Square Yard Plot.

To avoid this difficulty, Beaven invented in 1909 the "square yard" plot, which is formed by sowing eight rows six inches apart, four feet long, and with seed two inches apart in the row. This gives in the first place a plot 4 feet by 4 feet; but at harvest the outside rows are rejected and the outside 6 inches at each end of all the other rows, thus leaving the inside square * yard for the measurement of yield free from the competition of other varieties.

So far as I am aware, no one has made any further enquiry as to the most economical size of plot; the square yard plot only utilises for yield determination 9/16 of the experimental area, and to make it smaller would waste still more ground, while the larger the plot the more we depart from the principle of maximum contiguity.

* There has been some controversy in America as to the advisability of testing varieties in alternate rows, but lately T. A. Kiesselbach (*Journ. Am. Soc. Agron.* 1919, No. 6, pp. 235—241, "Experimental Error in Field Trials"; pp. 242—247, "Plant competition as a source of error in Field plots") has come to much the same conclusion as Beaven, viz., that although certain varieties *may* not under some circumstances interfere with one another, yet it is dangerous to allow any chance of the experiment being subject to this source of error, and that the only safe thing to do is to surround each experimental area with a border of the variety grown upon it, and to discard this border at harvest.

There are probably not enough data to discover by the calculus the size of plot which will give the minimum probable error per acre, and no one seems to have faced the labour of an experimental determination. At all events, without any further investigation the square yard plot has been adopted as the unit in some six or seven experimental cages in the British Isles.

COMPARISON ON A "CHESSBOARD."

Having adopted the unit, it was a comparatively simple matter to set units of two varieties in a "chess" or "chequer" board: subsequently it was found that more than two varieties could be economically compared at the same time.

To illustrate the problems which arise when we come to compare several varieties grown together on a "chessboard," we may take Beaven's No. 1 Yield Experiment of 1913 *.

In this, twenty plots of each of eight races of barley were grown on a regular system of repetition, and the following observations were made for each plot:

Number of plants.
Number of ears.
Weight of ears.
Weight of straw.

For the purpose of this illustration we need only consider yield of corn, i.e., weight of ears.

The eight races consisted of:

- Four Strains of Archer:
  - English Archer ... } Selection made by Beaven.
  - 7A ... ... }
  - Irish or Early Archer } Selection made by Capt. H. Hunter, B.Sc., of the Irish Department of Agriculture.
  - Irish Archer, No. 5 }
- Plumage ... ... — A selection made by Beaven which originated in Denmark. Wide-eared barley somewhat like Goldthorpe.

Each of these was, of course, descended from a single seed a few generations back, and

- Three Hybrids:
  - 145, and 145/46 — From a Plumage-Archer cross made by Beaven, the second being a re-selection from the first.
  - "Biffin" ... — Selected by the Professor of that name from a Plumage-Archer cross of his own.

In order to simplify the comparison of errors it is best to work as long as possible, not with the standard error but the "variance," or square of the standard error. It has two advantages: (i) that variance can be added or subtracted without the preliminary squaring and subsequent extraction of the square root, and (ii) that the area required to give any required accuracy varies directly with it; in order to give the same error a comparison with a variance of 60 only requires half as much ground as a comparison with a variance of 120.

Further, the variance taken in each case will be the variance of the average of

* *Vide* Diagram I, p. 277.

20 plots or differences between plots, or whatever it may be, and to get this we divide by 19, and not by 20, to correct for the small number.

The following table gives the means and variances of the average of 20 plots for the eight races as follows:

TABLE II.

	Mean weight per plot, grammes	Variance of the average of 20 plots
145/46	318·7	94·7
Early Archer ...	306·5	138·9
7*A*	304·6	80·7
145	300·7	94·9
English Archer ...	297·8	128·8
Plumage	295·2	150·8
Irish Archer, No. 5	276·5	81·7
Biffin	270·8	142·0
Average	296·4	114·1

Correction for Position*.

There is a great disadvantage in correcting any figures for position, inasmuch as it savours of cooking, and besides the corrected figures do not represent anything real. It is better to arrange in the first place so that no correction is needed.

In the present case the "vertical" arrangement is satisfactory, but as to right and left it is not so. English Archer averages ·2 rows to the left of 145, ·4 to the left of 145/46 and so on, 1·4 rows to the left of Biffin. As the average value per plot of a row is about 3·3 grammes higher than that of the row on its left, it might be thought right to make the following corrections:

145/46 ...	318·7+1·0=319·7
Early *A* ...	306·5−0·3=306·2
7*A*	304·6−1·7=302·9
145	300·7+1·7=302·4
English *A* ...	297·8+2·3=300·1
Plumage ...	295·2+0·3=295·5
Irish *A*, No. 5	276·5−1·0=275·5
Biffin ...	270·8−2·3=268·5

* For an elaborate method of Correction for Inequality of Soil, see Pearl, "A Method of Correcting for Soil Heterogeneity in Variety Tests," *Journal of Agricultural Research*, Vol. v. p. 1039.

In this paper Dr Pearl has corrected yield on the analogy of a contingency table. The method, which is probably as good a way as any of correcting for position, seems to me to be open to serious objections. A blot on the paper is the publishing of a "probable error" calculated from four cases without either correcting for the very small number or calling attention to the fact that they are appreciably too low.

The error of a comparison would no doubt be reduced very slightly as it generally is by any operation of this kind.

In any case the order is not altered, and I do not think the correction is worth making; the proper course would have been to reverse the order of the plots half way through so as to compensate for a possible tendency to improve from one end of the experimental area to the other.

VARIANCE IN TABLE II.

With the small numbers in question the variance figures do not differ significantly, but incidentally there is no indication that the hybrids are more variable in yield than the pure lines.

In order to get a clear idea of what these figures mean, let us suppose that a standard error of 1% is desired, say 3 grammes, a variance of 9. That would require an area $\frac{114{\cdot}1}{9}$, or 12·7 times as large as the present 20 plots.

If now the plots had been randomly placed, the variance of a comparison between two of the races would have been approximately 228, and about 25 times as much ground as was used would have been required to reduce the standard error of a comparison to 1%.

In order to give a general idea of the nature of the variability, chiefly due to soil, which has to be regarded as error when we consider the yield of varieties, Diagram II has been prepared in which each 20 grammes of yield above 100 grammes below the average yield of the variety is represented by a diagonal line drawn across the square representing the plot. It will be noticed that the shading grows heavier towards the right of the diagram, and that while it is by no means regular, the correlation between the shading of neighbouring plots is obvious to the eye.

The arrangement of the different races in a chessboard is of course designed to take advantage of this correlation by comparing always neighbouring plots as in the following example which concerns the first pair of races in the table.

Beginning at the left hand of Diagram I, 145/46 is in the middle of the first vertical line, and Early Archer at the top—the former being indicated by the letter C, and the latter by E. The yield of the first is 265·6, and of the second, 230·1. That gives a positive difference of 35·5. The next appearance is in the third line, again a positive difference, this time of 44·4. In the third occurrence the 145/46 is in the fourth line, and the Early Archer in the fifth line, and the difference this time is negative and 37·4, and so on.

The variance of the average of the 20 differences thus obtained is 124·0, very much less than the 233·6, which is the sum of the variances of the averages of the two races.

Now, if there were only two races in the chessboard it would be comparatively straightforward—the standard deviation would be found from the variance, and Sheppard's Tables (or preferably with such small numbers, "Student's") would be

used to judge the significance of the mean difference. In point of fact, however, the two races do not stand alone, and the question arises whether it would not be better to take the average variance of all the 28 differences between all the possible pairs of eight races.

Of course it is not likely that all our races would have the same variance, but with our small numbers such differences as there may be are almost certainly swamped by the error of random sampling, which, as pointed out above, will account for the observed values. From that point of view then it is better to average.

Again, all the comparisons are not of equal value: Irish Archer No. 5 is always found exactly on the right of English Archer, while Plumage is either three squares above English Archer or two below and one row to the right, and as will be shown later, there are indications that this is enough to affect the variance. Still it is not a very big thing, and the advantages of using a single figure far outweigh the slight loss of accuracy. I have calculated the 28 variances and they range from 44·1 (English Archer—Irish Archer No. 5) to 192·9 (Early Archer—Plumage), with a mean of 107·9. This is slightly lower than the 114·1, the average variance of the races. In other words, we have gained by chessboarding to the extent that we are as accurate as if we had devoted twice the area to plots randomly arranged.

The calculation of these 28 variances is tedious, but fortunately there is a short cut which gives an identical result.

In the following proof capital subscripts indicate variance directly measurable, which is taken as the mean value of such variance, while small subscripts indicate variance deducible from the observations.

If we suppose the total variance σ_t^2 of mn plots (i.e. n groups of one of each of m races subject to the error of random sampling) to be divided into three parts:

(i) that due to the m races if measured without error...σ_r^2;

(ii) that due to the position of the n groups of m races from left to right of the diagram (in this case 20 groups of eight) also measured without error...σ_g^2;

(iii) the casual error, which is the only part subject to random sampling...σ_e^2;

these three parts may be assumed to be independent so that

$$\sigma_t^2 = \sigma_r^2 + \sigma_g^2 + \sigma_e^2;$$

also the variance of the means of the races as we measure them is

$$\sigma_R^2 = \sigma_r^2 + \frac{\sigma_e^2}{n} - \frac{\sigma_e^2}{mn},$$

the last term being due to the fact that we have only mn cases to give us the mean.

Similarly the variance of the means of the groups as we measure them is

$$\sigma_G^2 = \sigma_g^2 + \frac{\sigma_e^2}{m} - \frac{\sigma_e^2}{mn},$$

and the total variance as we measure it is

$$\sigma_T^2 = \sigma_t^2 - \frac{\sigma_e^2}{mn};$$

from which eliminating σ_t^2, σ_r^2, σ_g^2,

we find $$\sigma_e^2 = \frac{mn(\sigma_T^2 - \sigma_R^2 - \sigma_G^2)}{(m-1)(n-1)},$$

and consequently $$\frac{2\sigma_e^2}{n},$$

which is the variance of a comparison between n groups of two races, is

$$\frac{2m(\sigma_T^2 - \sigma_R^2 - \sigma_G^2)}{(m-1)(n-1)}.$$

* In my first attempt to obtain this formula, I overlooked the $-\frac{\sigma_e^2}{mn}$ in the three equations for σ_R^2, σ_G^2, and σ_T^2. It was only after receiving a letter from Mr R. A. Fisher, who had independently arrived at the correct formula, that I found my mistake. Mr Fisher sent me two proofs, one of which was purely algebraical, proving in his notation the identity

$$\frac{2}{m(m-1)} S \frac{1}{2(n-1)} \left\{ \sum_1^n (X_{pq} - X_{p'q})^2 - n(\overline{X}_p - \overline{X}_{p'})^2 \right\}$$
$$= \frac{\sum_1^n \sum_1^m (X - \overline{X})^2 - m \sum_1^n (\overline{X}_q - \overline{\overline{X}})^2 - n \sum_1^m (\overline{X}_p - X)^2}{(m-1)(n-1)};$$

and the other, which he himself prefers, I append below:

"Let there be n trials indicated by suffices 1..., q..., n of each of m varieties similarly indicated by suffices 1..., p..., m.

Recognising that not only differences of variety but differences in the conditions of the trials may have affected the yields, we may obtain an estimate of what the variability would be if the conditions of any one trial could be replicated in a number of experiments with the same variety, provided the following simple assumptions hold good. The yield obtained in any experiment is the sum of three quantities, one depending only on the variety; a second, depending only on the 'trial'; and a third, which may be regarded as the 'experimental error' varying independently of variety and trial in a normal distribution about zero with a standard deviation which it is desired to estimate.

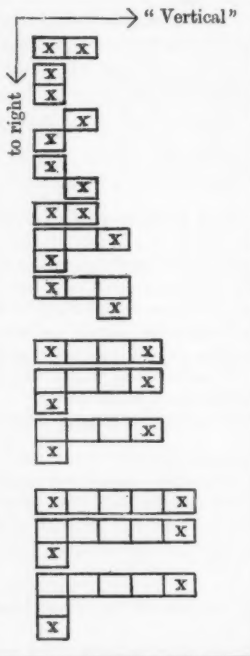
To obtain such an estimate we may fit the system of yields X_{pq} with a system of values $A_p + B_q$, choosing the latter so that

$$S(X_{pq} - A_p - B_q)^2 \quad \ldots\ldots\ldots\

To obtain the variance by this formula is a comparatively simple operation. In this case owing to the fact that I grouped the 160 observations in 10-gramme groups I got 109·3 by the short cut instead of 107·9, but it really should give an identical value.

Taking the square root we get a S.D. of 10·4 grammes or thereabouts for the standard error of a comparison, i.e., a probable error of about 2·4°/₀. This is probably as near as it is worth while going in any one season, for the experiment

DIAGRAM III.

Position	Distance between centres	Variance	Number of Differences
	4′	113·9	112
	4′	66·5	60
	5·7′	92·9	64
	5·7′	125·5	32
	8′	91·2	72
	9′	101·2	60
	9′	167·7	12
	12′	146·5	40
	12·7′	114·6	48
	14·4′	146·5	8
	16′	132·0	16
	16·5′	131·1	28
	17·9′	94·5	8

total. If either variety or 'trial' were without significant effect on the yield, the corresponding mean square would not differ significantly from that of line (*c*). To test the significance of such a difference we may use the fact that the estimates of variance in (*a*), (*b*) and (*c*) are all independent, and when m and n are fairly large the natural logarithm of the mean square has standard deviation $\sqrt{\frac{2}{n_1}}$, where n_1 is the number of degrees of freedom. In comparing two such independent estimates of the mean square, we therefore obtain the difference of their natural logarithms, and assign to it a standard deviation

$$\sqrt{\frac{2}{n_1}+\frac{2}{n_2}}.''$$

must be repeated several times to sample the weather properly, and cage area is too valuable to expend more than is absolutely necessary on a single experiment.

Before leaving this subject of chessboards, I would like to show in rather more detail that even with such small plots as these, slight differences in the arrangement within the group tend to increase the variance over that due to the ideal juxtaposition.

I have, therefore (see Diagram III, p. 284), separated the various kinds of comparisons and averaged the variance, in each case as that of the average of 20 differences.

The figures are not of course worth a great deal, but there is a marked tendency for the comparisons between the more distant plots to be the less accurate.

For purposes of illustration, I have correlated the distances with the variance for the 13 positions by the Spearman method, and get $\rho = +\cdot 41$.

THE HALF DRILL STRIP METHOD*.

The small scale work with which I have just dealt affords a means of picking out good varieties which can be tested in field trials. The whole eight varieties were tested on about 1/17 acre, sowing about a quarter of a pound of seed for each race. We now proceed to the most accurate method yet devised for field trials by which two varieties are compared on a total area of 5200 square yards, just over an acre, with, in the case which I shall give you, a standard error of ·63%. Of course, it will not necessarily be as low as this always.

The field is cultivated as usual up to the time of sowing, except that particular care is taken to clean the ground of weeds.

When sowing, the seed box of the drill is divided into two across the middle, and the middle coulter put out of action. The seed of the two varieties is put in the seed box, one on each side of the division. Thus when sowing a drill strip, one half (i.e. 6 or 7 rows) is sown with one variety and the other half with the other. On turning the drill at the end, the next strip is sown so that two half strips of the same variety are next each other, but care is taken to leave an interval between the two drill strips exactly equal to the gap in the middle of each drill strip between the two varieties. It requires careful steering but it can be done.

When the experimental field is sown, we get first a single half drill strip of one variety, then two of the other, then two of the first and so forth, ending with a half drill strip of the first. This ending is necessary in order to discount any fertility slope from one end to the other of the field. The space outside the experimental area should be sown all round with a similar grain, as the outside is naturally abnormal and is more liable to attacks from all kinds of enemies.

At harvest the outside row of each half drill strip next to the other variety is pulled up by hand and discarded to eliminate the "border" effect, and also to facilitate the use of the ordinary reaping machine. If the two varieties do not ripen

* For a full account *vide* "Trials of New Varieties of Cereals," by E. S. Beaven, *Journal of the Ministry of Agriculture*, Vol. XXIX. Nos. 4 and 5, 1922.

together one must be cut by hand when ripe, but if there is so little difference that both can be cut on the same day the reaping machine can be used on both. In either case each half drill strip is cut in such a way that the produce of each 1/500 acre can be tied up in two sheaves separately. In Beaven's case ten such 1/500 acre plots went to each half drill strip.

These sheaves can be weighed on the field, and so we can get the total produce of the field in plots of 1/500 acre and can compare each 1/500 acre with an adjoining one of the other variety.

Two things are to be noted at this point: (1) That without a very great deal of trouble the plots cannot be threshed out separately, but, fortunately, it has so far always been found where the matter has been put to the test that the variability of the yield of grain expressed as a percentage of the grain is less than the variability of the total yield expressed as percentage of total yield. In the Mercer and Hall experiment, the standard errors were 11·6°/₀ and 11·9°/₀, and Beaven's experience has been similar. Thus the figure which we obtain for the Standard Error is likely to be in excess of the truth. (2) From a practical point of view it is easier to work with a few half drill strips than a larger number of short ones, but if we depend on the weights of a few drill strips, there is considerable uncertainty about the Standard Error of the result. It was hoped that by determining the Standard Error of the difference between adjacent 1/500 acre plots, we could deduce the standard error of the average of n such differences by the formula $\sigma_a = \frac{\sigma}{\sqrt{n}}$, so that it would be immaterial whether the drill strips were long and few or short and many, as long as altogether there were n pairs of adjacent subplots. Indeed up to the time when I came to write this section, it was believed that this could be done. Beaven showed me his figures before publication, and I did not at the time observe that the formula cannot be used without further investigation, nor, so far as I am aware, has anyone else drawn attention to it. Nevertheless, I think it will be clear from the general considerations which have been advanced throughout the paper that there is a danger that the differences between corresponding constituent plots of a drill strip, even when they are as narrow as these, will tend to be correlated, and the formula $\sigma_a = \frac{\sigma}{\sqrt{n}}$, which requires independence of the individuals which are to be averaged, cannot be used without correction*. That this is so in the particular case which we are considering is made highly probable from the fact

* A fallacy arising from a similar neglect of correlation has come under my notice in some American work, but there the absurdity is more easily demonstrated. In the *Journal of the American Society of Agronomists*, Vol. IX. 1917, p. 138, A. G. McCall proposed that in order to save the trouble of harvesting and weighing 1/10th acre plots a number of square yards should be cut out and harvested separately, the square yards being taken systematically throughout the 1/10th acre plot, and the yield per acre calculated from these square yards. So far, so good, by taking enough square yards the slight loss of accuracy may perhaps be made up by gain in time or feasibility of operating. But in 1919, Arny and Steinmetz, *Journal of the American Society of Agronomists*, Vol XI. pp. 88, 89, applying this method, compared the error of the yield calculated from a few square yards cut from each of a number of 1/10th acre plots with that calculated from the 1/10th acre plots themselves. They found it substantially greater, but, say they, by increasing the number of square yards cut from each 1/10th acre plot to n, we

that the variance, expressed in terms of the percentage of the total weight of C, of the difference between the total produce from A and C is ·664 of the total weight of C when calculated from the 27 differences between adjacent half drill strips, while it is only ·301 when calculated from the 270 differences between adjacent subplots. The two figures should be the same within the error of random sampling, but differ probably by more than twice their S.D.

The results of the 1921 Trial are shown in Tables III and IV, which are taken, with his kind permission and that of the Ministry of Agriculture, from the Supplement to Beaven's paper, and give the weights of the sheaves, on the individual half drill strips, and on 243 of the 270 "plots," which go to make up the half drill strips respectively.

It will be seen that by taking the differences between adjoining half drill strips (or plots) a large part of the error is, as usual, eliminated.

Further, it is obvious that there is a general decrease in fertility as we go from drill strips with low numbers to drill strips with high numbers. It follows that the difference $A - C$ will tend to be greater when C follows A than when A follows C, and since this is always possible, experiments of this nature should always be planned so that there shall be an even number of differences, the series should begin and end with half drill strips of the same variety: in this case we may simply leave out the last drill strip and finish at half drill strip 52.

There is also a curious feature about these figures which can only be put down to some systematic error in technique; namely that when we compare together the adjacent half drill strips of A, that with the higher number always yields higher, although the general fertility runs the other way, and the same is true with regard to C in eight cases out of 13.

Both these kinds of error (that due to the general fertility slope and that due to the different fertility of odd and even half drill strips) are largely eliminated by Beaven's arrangement by which in alternate comparisons A follows C and C follows A and this can be made evident by adopting as unit not the difference between adjacent half drill strips but that between the sum of the two contiguous half drill strips of A and the sum of the two half drill strips of C which enclose them.

can decrease the error in the proportion $\frac{1}{\sqrt{n}}$, and so we can actually determine the yield more accurately by weighing up 10 or 20 square yards than by weighing up the whole half acre. It is rather surprising that they did not realise that there are 484 square yards in 1/10th acre, so that by taking 484 square yards they would be likely to be more accurate than if they took any lesser number and *a fortiori* tremendously more accurate than they would be if they took the same 484 square yards and called it 1/10th acre! Of course their formula also should be $\sigma\sqrt{\frac{1+(n-1)r}{n}}$, where r is the correlation between the yields on the square yards composing 1/10th acre plots, and not $\frac{\sigma}{\sqrt{n}}$.

The same fallacy has been used to extol the "rod row" method of determining yield, i.e., the method of cutting along the drill a row one rod in length to represent the yield of the plot from which it is cut.

TABLE III.

Warminster Field Variety Trial, 1921. *Half Drill Strip Weights, comparing:— Two races of barley, viz. "C" and "A." Area of each half drill strip* = 100 *sq. yds. Total area* = 2700 *sq. yds.* = ·56 *acre for each race. Showing total weight of sheaves on each half drill strip.*

Half drill strip		Weight of sheaves on half drill strip		Difference between "A" & "C"	Half drill strip		Weight of sheaves on half drill strip		Difference between "A" & "C"
Number		lb.		lb.	Number		lb.		lb.
"C"	"A"	"C"	"A"	"A" – "C"	"C"	"A"	"C"	"A"	"A" – "C"
1	2	165·4	164·6	−0·8	29	30	160·9	160·2	− 0·7
4	3	159·5	173·4	+13·9	32	31	153·2	164·3	+11·1
5	6	169·3	169·3	—	33	34	144·9	154·3	+ 9·4
8	7	179·8	174·9	− 4·9	36	35	147·7	158·6	+10·9
9	10	172·5	177·6	+ 5·1	37	38	142·4	143·0	+ 0·6
12	11	170·7	182·9	+12·2	40	39	138·7	143·6	+ 4·9
13	14	173·3	167·5	− 5·8	41	42	131·1	143·2	+12·1
16	15	166·1	178·5	+12·4	44	43	141·6	145·3	+ 3·7
17	18	174·5	170·3	− 4·2	45	46	145·0	150·1	+ 5·1
20	19	163·3	176·0	+12·7	48	47	155·4	154·0	− 1·4
21	22	166·0	159·1	− 6·9	49	50	151·1	149·3	− 1·8
24	23	161·2	168·7	+ 7·5	52	51	145·6	149·7	+ 4·1
25	26	169·3	164·2	− 5·1	53	54	146·3	158·5	+12·2
28	27	156·5	167·0	+10·5	Total		4251·3	4368·1	—
					Average		157·5	161·8	+ 4·3
					per cent.		100	102·7	+ 2·7

TABLE IV.

Showing weights of each "plot" (two consecutive sheaves), 9 plots (= 18 sheaves) on each half drill strip. Total number of plots on 27 half drill strips = 27 by 9 = 243 plots of each race (excluding end sheaves which are left out of account in this case in computing "probable error").

Nos. of half drill strips, "C" "A"	"Plot" Weights 2 sheaves. lb. "C"	"A"	A − C lb.	Nos. of half drill strips, "A" "C"	"Plot" Weights 2 sheaves. lb. "A"	"C"	A − C lb.	Nos. of half drill strips, "C" "A"	"Plot" Weights 2 sheaves. lb. "C"	"A"	A − C lb.	Nos. of half drill strips, "A" "C"	"Plot" Weights 2 sheaves. lb. "A"	"C"	A − C lb.
1 2	17·0	17·6	+ 0·6	3 4	17·4	16·3	+ 1·1	5 6	16·4	16·2	− 0·2	7 8	18·1	17·9	+ 0·2
	15·8	16·9	+ 1·1		16·6	16·0	+ 0·6		17·7	16·8	− 0·9		16·8	17·1	− 0·3
	14·2	15·0	+ 0·8		16·5	16·1	+ 0·4		13·0	14·8	+ 1·8		14·8	15·3	− 0·5
	14·2	13·8	− 0·4		15·4	13·2	+ 2·2		14·5	14·4	− 0·1		16·0	15·9	+ 0·1
	16·2	15·8	− 0·4		15·6	13·7	+ 1·9		15·2	15·8	+ 0·6		16·3	16·6	− 0·3
	17·3	17·0	− 0·3		18·9	16·7	+ 2·2		17·7	17·3	− 0·4		17·0	18·3	− 1·3
	18·0	17·1	− 0·9		18·6	15·6	+ 3·0		18·7	17·9	− 0·8		18·5	19·7	− 1·2
	17·8	17·2	− 0·6		17·8	16·9	+ 0·9		17·7	16·5	− 1·2		17·9	18·6	− 0·7
	16·6	17·8	+ 1·2		18·6	17·3	+ 1·3		19·1	18·9	− 0·2		19·3	19·7	− 0·4
	147·1	148·2			155·4	141·8			150·0	148·6			154·7	159·1	
	18·3*	16·4*			18·0*	17·7*			19·3*	20·7*			20·2*	20·7*	
	165·4	164·6	− 0·8		173·4	159·5	+13·9		169·3	169·3	—		174·9	179·8	− 4·9
9 10	17·6	16·7	− 0·9	11 12	18·0	16·1	+ 1·9	13 14	16·4	15·7	− 0·9	15 16	18·5	16·4	+ 2·1
	17·2	16·9	− 0·3		16·1	15·6	+ 0·5		16·5	17·1	+ 0·6		15·9	16·1	− 0·2
	16·0	15·8	− 0·2		15·9	14·2	+ 1·7		14·4	13·7	− 0·9		15·9	16·0	− 0·1
	17·0	15·9	− 1·1		16·9	15·6	+ 1·3		14·4	15·0	+ 0·6		16·0	16·0	—
	17·3	17·4	+ 0·1		18·1	16·5	+ 1·6		17·2	15·8	− 1·4		15·6	16·1	− 0·5
	16·0	17·2	+ 1·2		17·6	15·7	+ 1·9		16·5	17·3	+ 0·8		17·4	16·1	+ 1·3
	20·1	19·0	− 1·1		19·4	18·5	+ 0·9		19·6	16·8	− 2·8		19·6	16·7	+ 2·9
	18·6	18·5	− 0·1		18·2	17·9	+ 0·3		18·5	18·6	+ 0·1		20·3	17·4	+ 2·9
	20·5	19·0	− 1·5		19·2	19·9	− 0·7		21·8	17·3	− 4·5		20·1	17·1	+ 3·0
	160·3	156·4			159·4	150·0			155·3	147·3			159·3	147·9	
	12·2*	21·2*			23·5*	20·7*			18·0*	20·2*			19·2*	18·2*	
	172·5	177·6	+ 5·1		182·9	170·7	+12·2		173·3	167·5	− 5·8		178·5	166·1	+12·4
17 18	18·1	17·1	− 1·0	19 20	16·2	15·7	+ 0·5	21 22	16·3	15·9	− 0·4	23 24	14·4	15·2	− 0·8
	16·6	17·6	+ 1·0		17·7	15·2	+ 2·5		16·5	14·6	− 1·9		16·5	14·7	+ 1·8
	17·1	16·1	− 1·0		16·8	15·4	+ 1·4		16·4	13·6	− 2·8		15·3	16·2	− 0·9
	16·8	16·4	− 0·4		16·6	14·1	+ 2·5		14·7	14·0	− 0·7		14·6	14·2	+ 0·4
	15·8	14·6	− 1·2		16·8	14·6	+ 2·2		13·3	13·5	+ 0·2		14·3	14·6	− 0·3
	16·0	15·3	− 0·7		16·4	17·3	− 0·9		17·1	15·4	− 1·7		16·9	15·3	+ 1·6
	18·1	18·1	—		19·1	17·7	+ 1·4		18·2	18·1	− 0·1		18·7	17·3	+ 1·4
	17·3	18·0	+ 0·7		18·0	16·3	+ 1·7		16·8	17·2	+ 0·4		18·2	16·6	+ 1·6
	18·5	17·3	− 1·2		19·1	16·4	+ 2·7		16·9	16·7	− 0·2		18·7	17·0	+ 1·7
	154·3	150·5			156·7	142·7			146·2	139·0			147·6	141·1	
	20·2*	19·8*			19·3*	20·6*			19·8*	20·1*			21·1*	20·1*	
	174·5	170·3	− 4·2		176·0	163·3	+12·7		166·0	159·1	− 6·9		168·7	161·2	+ 7·5
25 26	15·5	14·1	− 1·4	27 28	14·7	14·6	+ 0·1	29 30	14·7	14·0	− 0·7	31 32	14·2	13·5	+ 0·7
	13·7	16·7	+ 3·0		15·6	15·3	+ 0·3		14·2	13·6	− 0·6		15·3	13·6	+ 1·7
	16·0	14·5	− 1·5		14·8	15·3	− 0·5		14·8	14·5	− 0·3		14·4	13·5	+ 0·9
	14·1	15·7	+ 1·6		14·2	13·7	+ 1·5		15·7	14·8	− 0·9		13·3	12·0	+ 1·3
	16·6	16·9	+ 0·3		15·4	15·8	− 0·4		16·9	15·9	− 1·0		17·1	17·3	− 0·2
	17·2	15·2	− 2·0		18·8	15·8	+ 3·0		16·4	16·5	+ 0·1		16·8	14·7	+ 2·1
	19·4	18·3	− 1·1		18·3	17·0	+ 1·3		17·0	16·8	− 0·2		18·9	17·3	+ 1·6
	18·0	17·1	− 0·9		19·2	15·7	+ 3·5		16·7	18·0	+ 1·3		19·0	15·9	+ 3·1
	18·6	16·8	− 1·8		17·5	16·6	+ 0·9		16·5	17·5	+ 1·0		17·2	17·8	− 0·6
	149·1	145·3			148·5	139·8			142·9	141·6			146·2	135·6	
	20·2*	18·9*			18·5*	16·7*			18·0*	18·6*			18·1*	17·6*	
	169·3	164·2	− 5·1		167·0	156·5	+10·5		160·9	160·2	− 0·7		164·3	153·2	+11·1
33 34	13·5	14·8	+ 1·3	35 36	15·0	13·1	+ 1·9	37 38	12·6	11·4	− 1·2	39 40	11·8	11·4	+ 0·4
	10·7	12·5	+ 1·8		13·7	11·3	+ 2·4		11·4	12·9	+ 1·5		11·9	11·1	+ 0·8
	12·8	12·7	− 0·1		12·6	13·5	− 0·9		11·9	11·9	—		12·3	12·9	− 0·6
	12·6	14·0	+ 1·4		14·3	14·5	− 0·2		15·4	14·3	− 1·1		12·8	13·6	− 0·8
	15·3	16·3	+ 1·0		16·4	16·8	− 0·4		15·1	16·7	+ 1·6		16·6	13·1	+ 3·5
	15·3	15·9	+ 0·6		17·4	12·6	+ 4·8		14·0	15·8	+ 1·8		14·2	15·3	− 1·1
	17·5	18·1	+ 0·6		15·3	17·4	− 2·1		15·1	13·8	− 1·3		16·8	16·0	+ 0·8
	15·9	17·5	+ 1·6		16·1	14·6	+ 1·5		13·8	14·7	+ 0·9		15·3	14·7	+ 0·6
	14·8	15·1	+ 0·3		17·9	16·3	+ 1·6		16·0	14·7	− 1·3		16·2	13·8	+ 2·4
	128·4	136·9			138·7	130·1			125·3	126·2			127·9	121·9	
	16·5*	17·4*			19·9*	17·6*			17·1*	16·8*			15·7*	16·8*	
	144·9	154·3	+ 9·4		158·6	147·7	+10·9		142·4	143·0	+ 0		143·6	138·7	+ 4·9

* These figures represent weights of the first and last sheaves on each half drill strip added together, and are excluded in calculating the average weights and also in calculating the "probable error."

TABLE IV (*continued*).

Nos. of half drill strips, "C" "A"	"Plot" Weights 2 sheaves. lb. "C"	"A"	A − C lb.
41 42	10·6	11·4	+ 0·8
	9·7	11·1	+ 1·4
	11·4	10·1	− 1·3
	11·9	10·7	− 1·2
	12·9	13·3	+ 0·4
	13·8	15·9	+ 2·1
	16·5	19·5	+ 3·0
	14·4	16·6	+ 2·2
	14·1	16·2	+ 2·1
	115·3	124·8	
	15·8*	18·4*	
	131·1	143·2	+12·1
49 50	13·6	13·8	+ 0·2
	12·5	13·0	+ 0·5
	13·6	13·8	+ 0·2
	14·5	13·0	− 1·5
	15·3	13·9	− 1·4
	15·9	15·6	− 0·3
	16·6	15·6	− 1·0
	16·8	17·1	+ 0·3
	16·4	16·0	− 0·4
	135·2	131·8	
	15·9*	17·5*	
	151·1	149·3	− 1·8

Nos. of half drill strips, "A" "C"	"Plot" Weights 2 sheaves. lb. "A"	"C"	A − C lb.
43 44	11·1	11·8	− 0·7
	10·9	12·0	− 1·1
	12·3	11·0	+ 1·3
	12·0	13·6	− 1·6
	14·4	15·6	− 1·2
	15·5	15·0	+ 0·5
	17·3	16·2	+ 1·1
	15·7	15·8	− 0·1
	18·3	14·3	+ 4·0
	127·5	125·3	
	17·8*	16·3*	
	145·3	141·6	+ 3·7
51 52	14·2	13·5	+ 0·7
	12·8	11·5	+ 1·3
	14·3	13·7	+ 0·6
	13·5	14·2	− 0·7
	14·3	14·1	+ 0·2
	15·3	14·1	+ 1·2
	14·5	16·0	− 1·5
	16·7	15·7	+ 1·0
	16·4	15·6	+ 0·8
	132·0	128·4	
	17·7*	17·2*	
	149·7	145·6	+ 4·1

Nos. of half drill strips, "C" "A"	"Plot" Weights 2 sheaves. lb. "C"	"A"	A − C lb.
45 46	11·6	13·2	+ 1·6
	11·7	12·5	+ 0·8
	10·7	12·2	+ 1·5
	13·9	14·5	+ 0·6
	15·8	15·6	+ 0·2
	14·8	16·2	+ 1·4
	16·8	16·1	− 0·7
	17·4	16·0	− 1·4
	16·8	16·2	− 0·6
	129·5	132·5	
	15·5*	17·6*	
	145·0	150·1	+ 5·1
53 54	13·8	13·1	− 0·7
	12·3	12·2	− 0·1
	11·8	14·8	+ 3·0
	13·9	13·6	− 0·3
	16·2	15·6	− 0·6
	15·1	16·5	+ 1·4
	15·5	16·0	+ 0·5
	15·4	19·0	+ 3·6
	17·2	16·5	− 0·7
	131·2	137·3	
	15·1*	21·2*	
	146·3	158·5	+12·2

Nos. of half drill strips, "A" "C"	"Plot" Weights 2 sheaves. lb. "A"	"C"	A − C lb.
47 48	11·8	12·9	− 1·1
	13·0	13·4	− 0·4
	13·9	13·9	—
	16·7	14·9	+ 1·8
	15·6	16·3	− 0·7
	16·4	17·0	− 0·6
	15·6	17·0	− 1·4
	17·6	15·7	+ 1·9
	16·3	17·1	− 0·8
	136·9	138·2	
	17·1*	17·2*	
	154·0	155·4	− 1·4
Average weight per "plot" (end sheaves excluded)	15·56	15·88	+0·32
%	100·0	102·1	+2·10

* These figures represent weights of the first and last sheaves on each half drill strip added together, and are excluded in calculating the average weights and also in calculating the "probable error."

This may be described as a "sandwich," and it may be noted that just as there are subplots composing a half drill strip, so there are "sub-sandwiches" which will also tend to eliminate the same errors as the "sandwiches."

The following table gives the differences $A - C$ for the thirteen "sandwiches" composed of half drill strips 1 to 52:

TABLE V.

Half drill strip numbers	A − C	Half drill strip numbers	A − C
1 to 4	+13·1	29 to 32	+10·4
5 „ 8	− 4·9	33 „ 36	+20·3
9 „ 12	+17·3	37 „ 40	+ 5·5
13 „ 16	+ 6·6	41 „ 44	+15·8
17 „ 20	+ 8·5	45 „ 48	+ 3·7
21 „ 24	+ ·6	49 „ 52	+ 2·3
25 „ 28	+ 5·4		

The mean $A - C$ for sandwiches is + 8·05 and the variance, making allowance for the pitifully small number, is 51·41. This leads to a variance of the difference between the total produce of A and of C expressed in terms of the total weight of C of ·398, intermediate between the ·664 calculated from the half drill strip differences and the ·301 calculated from the subplot differences.

It should be noted at this point that the "sandwich" is a perfectly legitimate device for eliminating errors common to both variants whose difference is to be measured, and that it is only by using it that we can get the true value of the error of the comparison, whereas the subplot difference would really lead to a larger value than ·301 if we had sufficient knowledge to be able to apply the true formula

$$\frac{\sigma^2(1+(n-1)r)}{n}.$$

A similar calculation based on the "sub-sandwiches," i.e. sandwiches one plot in depth, gives a value of the variance ·248 corresponding to the ·398 from the whole sandwiches. The S.D of these to some extent correlated figures is not easy to determine but the difference between them must be of the order of once the S.D. This is not significant but with our small numbers it is not inconsistent with the expected correlation between the sub-sandwiches composing a sandwich. Until a number of experiments have been carried out in several places and the results submitted to analysis, it would be wise to keep the number of drill strips as large as possible and economise in length in spite of the practical difficulties of doing so.

Since the variance calculated from the drill strip sandwiches is subject to a large error of random sampling owing to the necessary paucity of numbers, it is well to calculate also from the "sub-sandwiches" and take the larger of the two in determining the standard error.

It is possible that some of my readers may devise some better method of utilising the weights of the "subplots" than I have been able to do, and I commend the problem to them.

In the present case it is probably better with only thirteen sandwiches to take the standard error of a single sandwich and use "Student's" tables, when the probability that such a large positive difference should occur by chance is found to be ·001. The difference is therefore quite significant. If however it is required to compare the standard error with other experiments, we can say that the most probable value is only ·63 % on a total area of about 1 acre.

Other precautions such as correction for moisture, etc., are taken as a matter of course.

CONCLUSIONS.

The chief difficulty of comparing varieties consists in the fact that the differences to be measured are quite small compared with the variations due to soil and weather. While the latter is not within our control, the errors due to the soil may be reduced to reasonable proportions in any one of three ways:

(1) Large plots may be repeated many times. An instance is given of this when in the Irish two-acre experimental plots a difference of 7 % in the value per acre was proved with a standard deviation of about 2 % in 51 trials, extending over six years.

Undertakings of this magnitude are hardly to be put in hand by any but Departments of State.

(2) Quite small plots of one square yard, surrounded by a border of the same variety as in the square yard, may be grown under a wire cage on a regular system, technically called a "chessboard." An instance of this is given when, in Beaven's No. 1 Yield Experiment of 1913, eight varieties were compared on a total area of about 1/17th acre using about 5 oz. of seed of each variety, with a standard deviation of a comparison in a single year of about $3\frac{1}{2}$%.

The large number of varieties which may be compared at once, and the small area which is required, make this an ideal method of testing new varieties. On the other hand, a wire cage is not a cornfield, and the varieties found to be best in the cage will always require further testing on the large scale. The method is, however, within the powers of anyone who can build a cage, and has the necessary skill and patience to conduct the experiments.

(3) By means of Beaven's "half drill strip" method, two varieties may be compared on a total area of about one acre in one year with a standard deviation of a comparison of less than 1%. This combines the advantage of growing corn on the large scale with an accuracy almost as great as that of small scale work; and is within the powers of anyone who can combine the necessary knowledge and patience with the control of skilled agricultural labour.

It is shown that methods (2) and (3) depend for their accuracy on the fact that the nearer two plots of ground are situated, the more highly are the yields correlated, so that we are able to increase the effect of the last term of the equation

$$\sigma^2_{A-B} = \sigma^2_A + \sigma^2_B - 2\, r_{AB}\, \sigma_A\, \sigma_B$$

(where A and B are the varieties to be compared) by placing the plots to be compared with one another as near together as possible.

A formula, due to Mr R. A. Fisher, is given for calculating the error of a comparison in a "chessboard" experiment, which may perhaps be found useful elsewhere.

Finally I have to thank Dr Beaven both for allowing me to use his experimental material and for much invaluable assistance in the preparation of the paper.

Addendum.

Since writing the above I have had the advantage of witnessing the harvesting of Dr Beaven's 1923 experiment and of discussing the whole question with him very thoroughly.

He thinks it probable that the whole or a part of the correlation between the yields of the "plots" which together formed a drill strip in the 1921 experiment may have been due to slight differences in area consequent on irregular steering of the seed drill, such as would have been caused by the horses pulling unequally.

Measurements which we made on the stubble of the similar 1923 experiment showed not only that such inaccuracies occur, but also that they can favour one of the varieties.

It is however a fairly easy matter after harvest to measure the total width from the outside drill of one half drill strip to the outside drill of the same variety. This measurement includes the space between the drill strips which is variable owing to the difficulty of steering and is now made in practice across each drill strip in several places.

It is thus possible to estimate accurately the total area occupied by each variety and to make the necessary correction to the total yields.

As however it would hardly be possible to correct the individual drill strips or "plots" which are used for the purpose of calculating the error, that calculated error will be in excess of the truth.

In Dr Beaven's opinion the operation of taking differences has for all practical purposes eliminated the correlation due to the position of the "plots," and in view of the other causes of variation in the differences, numerous and diverse as they are, he still considers it legitimate to treat the differences between the "plots" as if they were random, and to use the formula $\frac{\sigma}{\sqrt{n}}$ in calculating the error of his mean difference. I feel however that a single operation of this nature is hardly likely to eliminate all the correlation and that there is need for further enquiry: if as the result of a number of experiments it is found that the error of the mean difference calculated from the weights of the half drill strips is not significantly greater than that calculated from the "plots," then the latter undoubtedly provide the more accurate data for the calculation of that error, and it will be a matter of indifference whether the drill strips be few and long or short and many.

Meanwhile they should be made as numerous as is consistent with the successful carrying out of the various agricultural operations which are of course made infinitely more difficult and tedious by the necessity of turning horses and machines at the end of each short length.

But whether we use few long or many short strips is not a question of the first importance: in either case the method is without doubt the best that has hitherto been devised for large scale experiments.

ON THE NEST AND EGGS OF THE COMMON TERN (*S. FLUVIATILIS*). A THIRD COOPERATIVE STUDY.

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J. O. IRWIN, B.A. AND KARL PEARSON, Computers and Reporters*.

CONTENTS.

PAGE

(1) Introductory 294
(2) Report on Measurement and Observation of Terns' Eggs and Nests, 1920 295
(3) Characters Measured or Observed 301
(4) Influence of Breadth of Egg on Mottling and Ground-Colour . . . 303
(5) Change of Type of Egg with Season 306
(6) Association of Type of Nest and its Environment with Egg Characters . 308
(7) The Organic Correlations in Size and Shape 317
(8) The Organic Correlations of Mottling with Ground-Colour . . . 318
(9) Cross Organic Correlations between Pigmentation Characters and Size and Shape Factors 320
(10) The Homotypic Correlations
(i) Direct Homotyposis in Size and Shape 324
(ii) Direct Homotyposis in Pigmentation 325
(iii) Cross Homotyposis in Size Characters. 327
(iv) Cross Homotyposis in Pigment Characters 328
Appendix: Tabulated Data 330

PLATES

Plate I. Colour Scale for Ground-Colour, and Special Eggs selected for Shape and Pigmentation . . . . . *to face p.* 301
Plate II. Mottling Scale . . . . . „ 301
Plate III. Illustrations of Environment and Make of Nests . . „ 302
Plate IV. Blakeney Point, the Sand Dunes, Far Point, and a Nest „ 302

(1) *Introductory.* This paper is a third cooperative study of the nests and eggs of the Common Tern, the two earlier ones corresponding to the layings in 1913 and 1914 respectively and this to that of 1920. There are, however, very considerable difficulties about the comparison of the earlier years and 1920. In the first place the war years intervened and the presence for several years of soldiers at Blakeney Point altered very considerably the environmental conditions of the colony, the eggs being considered a source of food-supply. Further in 1920 the fieldworkers lost the assistance of Mr William Rowan, and with the exception of Dr Katherine Watson, who had assisted Mr Rowan in 1913, were all fresh to the

* A certain amount of the laborious tabling was undertaken by Senior Honours Students working in the Biometric Laboratory, to whom our thanks are due.

work. They were, as many previous workers had been, dependent on Mr R. J. Pinchen, the Bird-watcher at Blakeney Point, for the identification of most of the birds seen or heard; they owe much to his unrivalled knowledge. New fieldworkers mean different personal equations, and new categories with a view to increased accuracy of observation were introduced, so that occasionally direct comparison with earlier reductions is not easy. It is always a grave question in the case of any census how far it is advisable with the hope of increased accuracy to adopt new categories of classification. Unless the new categories are merely subdivisions of the old, it becomes very difficult to measure secular or seasonal changes.

In the 1920 observations more elaborate schemes of mottling and ground-colour were adopted, from which it is difficult to reproduce the simpler schemes of 1913 and 1914. Mr William Rowan was very sure that all the ground-colours of eggs of the earlier years could be classed into pure brown or pure green of various shades of intensity*, and this is in keeping with his paintings of eggs. In 1914 60 °/₀ of eggs were recorded as of green, 40 °/₀ as of brown ground-colour. Examining the scale adopted in 1920, an approximately accurate copy of which is provided in the centre of Plate I of this memoir, it is very hard to determine which shades would in 1914 have been selected as definite greens. Clearly 1, 2, 3, 4 are such. 1*A*, 2*A*, 3*A*, 4*A* and less certainly 1*B*, 2*B*, 3*B* and 4*B* may be said to have a green hue. Omitting 9 exceptional eggs, this gives us 531·5 eggs out of 947 or 56 °/₀ as green. We should have to include some of the *C*'s, say C_4 with 37 eggs, to reach something like 60 °/₀ of green eggs. But such a classification into greens and browns would be very risky. A diagonal division into greens and browns gives us 576 eggs out of 947 or 61 °/₀ green†. This is in fair accordance with the 1914 value, but leaves us quite in the dark as to which of the individual eggs recorded as 1*A*, 2*B*, 3*C* and 4*D* are to be treated as green and brown respectively. Besides which objection, it disregards Mr Rowan's strongly expressed opinion that all eggs can be classified as either brown *or* green. The 1920 fieldworkers were very distinctly opposed to this view. The absence of Mr Rowan from this country has prevented a discussion of the matter between the two sets of fieldworkers, and accordingly very little comparison is possible between the earlier and later work in ground-colour‡.

Much the same remarks apply to mottling for which a new and more elaborate scale was introduced; it again admits of little comparison with the earlier scale. The nature of the new scales will be indicated in the following extracts from the Report prepared by Dr K. M. Watson in 1920§.

(2) *Report on Measurement and Observation of Terns' Eggs and Nests, June 14th to July 1st*, 1920. The eggs of the common tern were this year abundant and the

* Omitting of course exceptional blues. See *Biometrika*, Vol. XII. p. 318 ftn.

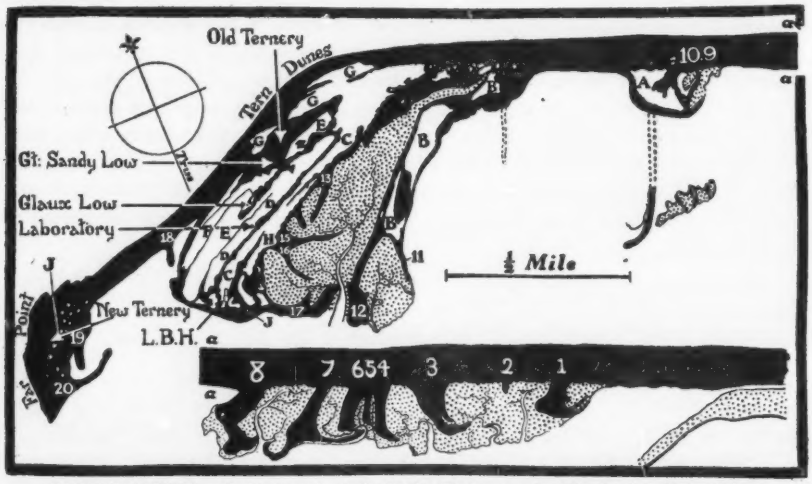
† Halving the contents of the diagonal categories.

‡ Naturally the points discussed on pp. 318—325 of the second paper (*Biometrika*, Vol. XIV.) must be considered in suspense.

§ Since more completely published in the *Transactions of the Norfolk and Norwich Naturalists' Society*, Vol. XI. pp. 168—179, 1920—21. The original report was read and approved by Professor D. M. S. Watson, M. N. Karn and H. S. Pearson, and various additions were made owing to their suggestions.

fieldworkers measured 412 clutches represented by more than 900 eggs. When we arrived on June the 14th a number of chicks had already hatched out and many of the remaining eggs were cracking. About June 20th the birds started laying again so that we were able to measure some eggs of the second batch, but the birds were still laying on July 1st when we left. Pinchen assured us that we had missed hundreds of eggs by coming so late and leaving so soon. The majority of the nests were in the neighbourhood of the Tern Dunes, some actually amongst the embryo dunes, many on the shingle immediately to seaward of these dunes and a fair number in Great Sandy Low. Another big batch of nests was situated on the new shingle beach, isolated nests being found on most of the minor spits, a good number on the large Far Point and a few scattered all along the ridge joining this beach to the shingle ridge in front of the Tern Dunes*. A small colony of nests was found at the foot of the beacon hills on the seaward side

* See map below. Professor F. W. Oliver, to whom it is needless to say that this investigation like the previous papers owes much, writes: "In the matter of environment things are changing. In 1920 the birds were beginning to supplement their old ground (Great Sandy Low) by using what we call the Far Point. Perhaps a third to a quarter of that year's data was derived from the new area. To-day more than half the birds use the New Ternery (cf. Plate IV). It is more exposed and wind-swept, but has tidal waters near at hand on both sides. Year by year the vegetation here advances by leaps and bounds. For 1923 I have charts of parts of the vegetated area with the position of the nests marked thereon. Gradually new species of terns have come and increased, including Arctic, Roseate and Sandwich Terns. Of the last named there were perhaps 200 nests in all. The Sandwich Tern builds no nest, but lays the clutch *always* on bare sand adjoining embryo dunes, and so close are the nests together that you may find 50 nests on an area not greater than the floor of an ordinary room." [Letter of F. W. O. to K. P. Oct. 5, 1923.]



BLAKENEY POINT: scale, 3 inches to 1 mile (1 : 21,160). The lower section (Cley Beach to Marrams) fits at *a*, *a*, on to right-hand end of upper section. *Shingle beaches*, where exposed, are marked in black, the laterals (or "hooks") being numbered in their supposed developmental sequence. *Dunes* are white, successive systems being lettered A to J. The fringing dunes H, alphabetically out of position, represent a S.E. extension of the Headland series, from which they are derived by strong N.W. winds. In *time* they are subsequent to the G system. *Salt marshes* are dotted. The youngest to establish (since 1910) is the detached area ⅓ mile south of The Hood (A). Adapted from air survey photographs of 1921.

about level with the lifeboat house. Two or three odd nests were found about 100 feet west of telegraph poles 90 and 91, two in the east end of Glaux Low and several on the fairly high dunes on the seaward side of Glaux Low at its point of junction with Great Sandy Low. A number of the nests in Great Sandy Low and in the Tern Dunes were robbed by rats, and according to Pinchen this caused the birds to start laying again. Much to his regret they repeated their original indiscretion of laying near the dunes and in some cases we saw the second nests emptied by rats. The second laying gave us the opportunity of collecting some data as to the laying habits and method of nest building, as we were able to watch individual nests from day to day and thereby we obtained some information to supplement the observations of Mr Rowan on this subject.

Colour Scale. After a preliminary tour of the nesting ground conducted by Pinchen on the morning of June 15th we proceeded to devise new colour and mottling scales. The method by which the colour scale was obtained was as follows:—A two-and-a-half-inch square of white paper was divided into half-inch squares. Two colours, one representing as closely as possible the lightest brown type of "brown" egg, the other the lightest green type of "green" egg, were obtained. One line of squares 0—4 (see Plate I) was left white and the brown colour washed over the remainder of the squares once. This wash was allowed to dry and the green was applied to all the squares except 0—*D*, i.e. those above the square 0, which was originally left white. One square (0) was thus left white, one line *A*, *B*, *C*, *D* pure brown, one 1, 2, 3, 4 pure green, the rest mixed. The green wash having dried, another coat of brown was applied to the three rows of squares farthest from the green line (rows *B*, *C*, and *D*). Another green wash was then applied to the three rows farthest from the brown row (rows 2, 3, and 4). This process was carried out so that eventually we had four shades of pure brown (*A*—*D*), four of pure green (1—4) and sixteen squares of mixed colour containing all the possible combinations of the four green and four brown shades. Small square holes were cut in each individual square so that the eggs could be held behind the paper for purposes of comparison. By this method we obtained a colour scale into which all the normal eggs could be fitted with a fair degree of accuracy, but the scale would be improved by carrying on the pure brown and pure green lines (particularly the brown) so as to obtain a darker shade of brown and green without mixture. The need for this is evident from notes on the colour of an egg such as "*D* but darker" or "4*D* but much browner."

Another weak point in our colour classification arises from the fact that two scales had to be made for the two pairs of workers and although both were done with the same washes on the same paper by the same method, the colours differed slightly. A table of equivalent colours was made, but it was found that some of the colours on one scale could not be accurately matched on the other. It should be noted that nearly all the colour estimations of eggs were done in full bright sunlight.

Mottling Scale. The method by which the type of mottling was determined was as follows: two main factors were observed to be concerned. (1) The size of

the markings; (2) Their distribution. (It was not found possible to estimate the amount of mottling relative to the surface of the egg except in the inaccurate way of noting exceptionally densely or sparsely mottled eggs.) Two series of types were therefore made, one for coarse, the other for fine markings, each containing four types of distribution, viz.: (1) Distribution even; (2) Slight concentration in a belt; (3) Marked concentration in a belt; (4) Concentration at the pole. In addition to the eight types thus made three rather aberrant types were found and put on the scale, viz.: (1) Few coarse blotches irregularly scattered; (2) A dense continuous (or almost continuous) belt with few markings elsewhere, and (3) Very few, very large blotches. A 12th type of coarse markings with spiral twist was based on an egg figured by Mr Rowan but this exact type was not found on this occasion. A spiral twist of the marks may occur in almost any type of mottling and was noted in each case. In practice we found the greatest difficulty in deciding whether the markings on an egg were to be regarded as coarse or fine as the types grade into one another so completely. Furthermore, the so-called equatorial belt may occur at almost any level on the egg, and a series of types based on the position of the belt could be made. It is thus evident that the existing scale for classifying the mottling is by no means satisfactory.

The colour of the dark spots which form the mottling is always some shade of brown. Three types seem to occur: (1) Not very dark greyish brown splashes usually of medium size, which appear to be buried in the substance of the egg shell. The colour of these splashes becomes lighter towards their margin. (2) Dark brown spots which lie on the surface and have sharply defined margins. These form the bulk of the mottling of most eggs. (3) Very small absolutely black spots, seen only very rarely.

The type of nest was recorded by a brief description of each nest and the general colour of the surroundings was also noted.

A new method of measuring the longitudinal and transverse girths was used. A number of strips of thin paper were cut, and the girths were measured by wrapping a strip round the egg, making a mark with a pencil on the strip, and measuring with a scale.

The eggs varied in length from 3·70 cms. to 4·68 cms. and in breadth from 2·69 cms. to 3·28 cms. In a certain number of cases the equators of eggs were found to be elliptical instead of circular, and this was noted.

As regards colour a few eggs were found which would not fit into our scale. The most striking of these were blue eggs. One nest was found in Great Sandy Low containing 3 pale grey-blue eggs with dark brown markings. In addition to this nest, several odd blue eggs were found in nests with eggs of other colours. Amongst the most striking of these nests are the following: 176 (*a*) Gd. col. 3*B*, (*b*) pale blue with few faint coarse markings; 325 (*a*) 3, but brighter, (*b*) light blue; 326 (*a*) 3, (*b*) bright blue, mottling all over but concentrated on top and very fine; 342 (*a*) about density of 2 but perfectly blue, (*b*) 2. Of these nests 176 was situated on the drift line on the new beach, 325 and 326 on shingle, these three

being about as far as possible from any living plants, 342 and 91 which contained two bluish eggs were on the shingle, near the Tern Dunes. A number of eggs were found which were too yellow, and others too grey, to fit accurately into the scale. We noted that the eggs when first laid had a peculiar brightness; in several cases when we observed nests for a second time some days after making our original measurements we found that the eggs had faded somewhat. An interesting case was shown by nest 197, which contained two eggs of different colours, (*a*) 4*D*, (*b*) 2*B*, but both eggs were marked with slight ridges running round the egg horizontally. This was the only case in which such markings were seen and it may therefore be taken as certain that these two eggs were laid by the same bird.

Construction of Nests. As usual there was considerable variety in the materials used for building nests. A few nests were made exclusively out of shells; dried starfish, cast crab shells, parts of skeletons of birds, a small dried cuttle-fish were amongst the curious objects found in nests. Dried Psamma and fine twigs were the most popular nesting materials. In several cases where we watched a nest from the time of laying the first egg, we found that additions were made to the nest subsequent to that event, e.g. nest 24 when first observed (on June 16th at latest) had "one piece of Psamma, one stick, and one or two short sparse tufts of grass." Later on June 27th it has become "a very well made nest, lots of Psamma, sticks and straws circularly arranged and padding nest well, a long feather, several small chips of wood, a small chunk of drift, several blobs of tar." Two eggs had been laid in the interval. Our observations went to show that the type of nest built depended on the availability of material rather than on any other factor. For example, the nests on the new beach where materials were scarce mostly contained only a few chips, whereas the nests near the dunes where materials were abundant were often most elaborate, whilst the nests actually in the lows and on the dunes were made for the most part of the ubiquitous Psamma.

With regard to the number of eggs in the clutches, we are all of opinion that a genuine one-egg clutch occurs seldom if ever. In most of the cases where we found only one egg in a nest, it was either the last egg to be hatched and we found chicks in or near the nest, or it was the first egg laid, or the other egg or eggs had been stolen, or it was an addled and deserted egg. A certain number of genuine two-egg clutches do occur but the number of these is smaller than would appear from our records, because many apparent two-egg clutches are produced in the same ways as the apparent one-egg clutches. The majority of the nests certainly originally contained three eggs, that being the normal clutch for the common tern. One nest (309) was found with four eggs in it, but one of these (*a*) was addled and broken. A few days later the nest was revisited; a fifth egg had been laid and the broken egg had been ejected. We concluded that a bird had started laying again in an old nest and that eggs *a* and *b* which resembled each other belonged to a previous clutch and *c*, *d* and *e* were probably new.

Various Vicissitudes. A considerable number of eggs were found which had been washed out of their nests by a high tide. For the most part it was found

useless to measure these, but in a few cases where we found two or three eggs together apparently representing a genuine clutch, we measured them, but always recorded the fact that they had been washed out.

The mortality amongst the young chicks appeared to be fairly high. A good number of birds die whilst they are trying to get out of the egg, so that a considerable number of eggs were found chipped and with the beak of a dead bird protruding. A good many both eggs and small chicks fall a victim to the rats, in spite of Pinchen's untiring efforts. Gulls destroy a certain number of eggs. The cause of death in so many small chicks was not at all evident but certainly only quite a small percentage of the eggs laid ever produce mature birds.

The terns showed themselves as fearless as ever, sitting on their nests or feeding their chicks within a few yards of us when we were at work. We learned to recognise some individual birds by their peculiar ferocity, and, for the first time on record, several birds did actually attack the workers, each of whom was at some time pecked on the head. On one occasion two of us happened to get a full view of the tern colony on the far beach mobbing a gull and it became evident that if the birds chose to cooperate in attacking a human being as they do in rejecting an intruding gull, we should soon be driven off the ground. Another instance of the value of cooperation was seen when several enterprising baby-terns took it into their heads to swim out to sea. A group of old birds by their united efforts succeeded in chivying them back to safety on shore. The greater part of the time, however, the birds seemed to be as intolerant of each other as they were of us, their constant calling being generally intermingled with the yet more raucous sounds of squabbling.

Note on the Laying Habits of the Common Tern. As has been mentioned above excellent opportunities were afforded during the second breeding season in 1920 for making observations on the intervals between the laying of the first, second and third eggs of the common tern. Mr Rowan stated in his report for 1914 that the bird which he photographed laid her three eggs on three successive days. She appears to have been somewhat exceptional in this as the interval in most cases which we observed this year was longer than twenty-four hours. On June 21st when going round Great Sandy Low, M. N. Karn and the writer found several nests with only one egg which appeared to us to be newly laid. We made special notes of these nests and from that day until June 27th we went round Great Sandy Low every afternoon noting new eggs and nests.

Although we went over the areas in which the birds were nesting very carefully, we often found that we had missed nests completely, so that it is only in one or two cases that we feel convinced that a nest was new on the date on which we first found it. When we had spotted a new nest however, we always marked it in some way, so that our records for the second and third eggs are fairly accurate. As a rule we did not visit the nests more than once a day, though on one occasion we went at 10 p.m. as well as at 3.30 p.m. The following nests were observed in this way:

145. June 22nd *a*; 23rd *a* only; 24th *a* and *b*; 25th *a* and *b* only; 26th *a*, *b* and *c*.

169. June 21st *a* only; 22nd *a* and *b*; 23rd *a* and *b* only; 24th *a*, *b* and *c*.

77. June 21st *a* only; 22nd *a* and *b*; 23rd *a* and *b* only; 24th *a*, *b* and *c*.

186. June 22nd *a* only; 23rd *a* only; 24th *a* and *b*. No third egg laid by June 27th.

153. June 21st *a* and *b*; June 22nd *a*, *b* and *c*.

168. June 21st *a* only, at 3.30 p.m.; *a* and *b* at 10 p.m.

304. June 25th *a* only; 26th *a* and *b*.

301. June 25th *a* only; 26th *a* and *b*.

184. June 23rd *a* only; 24th not examined; 25th *a*, *b* and *c*.

305. June 25th *a* only; 27th *a* and *b*.

291. June 25th *a* only; 26th *a* and *b*; 27th *a* and *b* only; 28th *a*, *b* and *c*.

Nests 301 and 304 were almost certainly new on the days on which they were first recorded, for in one case Pinchen and in the other we had been over the ground very carefully on the previous day. In both these cases, then, the bird appears to have laid an egg on two consecutive days. The only case in which a bird appeared to have laid an egg on three consecutive days was that of nest 184 in which one egg was found on June 23rd, and three eggs on June 25th. Even this may not be a genuine case, for there is no proof that the nest was new on June 23rd. It therefore seems that the laying period is generally spread over four or five days.

(3) *Characters Measured or Observed.* There were those of 1913 and 1914 with the addition of Environment of Nest. Namely:

(α) Length (L); (β) Breadth (B); (γ) Longitudinal Girth (G_l); (δ) Transverse Girth (G_b)—taken as in the earlier years, except that paper strips were used to determine the girths; (ϵ) Ground-Colour; (η) Mottling;—both determined by new scales as described by Dr Watson in the preceding section. The Ground-Colour Scale is reproduced on Plate II. The Mottling Scale is provided on Plate II and has been indicated in the Tables by the following letters:

Fine Markings: (*a*) Even distribution; (*b*) Slight concentration in belt; (*c*) Marked concentration in belt; (*d*) Concentration at broad end.

Coarse Markings: (*e*) Even distribution; (*f*) Slight concentration in belt; (*g*) Marked concentration in belt; (*h*) Concentration at broad end.

Special Markings: (*i*) Few coarse blotches, irregularly distributed; (*j*) Dense equatorial belt; (*k*) Very few large blotches; (*l*) Coarse markings spirally twisted.

As already noted (*l*) was not used as it was found that a spiral twist of the markings could occur with any type.

Turning again to the colour scale we can make five shades of each colour:

(i) No Brown: i.e. 0, 1, 2, 3 and 4; say, 0.

(ii) Slight Brown: *A*, 1*A*, 2*A*, 3*A* and 4*A*; „ *A*.

(iii) Light Brown: *B*, 1*B*, 2*B*, 3*B* and 4*B*; say, *B*.
(iv) Medium Brown: *C*, 1*C*, 2*C*, 3*C* and 4*C*; „ *C*.
(v) Dark Brown: *D*, 1*D*, 2*D*, 3*D* and 4*D*; „ *D*.

and again

(i) No Green: 0, *A*, *B*, *C*, *D*; say, 0.
(ii) Slight Green: 1, 1*A*, 1*B*, 1*C*, 1*D*; „ 1.
(iii) Light Green: 2, 2*A*, 2*B*, 2*C*, 2*D*; „ 2.
(iv) Medium Green: 3, 3*A*, 3*B*, 3*C*, 3*D*; „ 3.
(v) Dark Green: 4, 4*A*, 4*B*, 4*C*, 4*D*; „ 4.

Assuming that the brown and green are different pigments (for example, melanin and lippochrome) we should thus have two distinct scales involved, and any egg may be recorded on both of them.

A third scale might be obtained by summing "values," in which case we should add the total number of washes in each category without regard to green or brown. Thus we have:

(i) No wash $= 0$; say, w_0.
(ii) One wash $= 1 = A$; „ w_1.
(iii) Two washes $= 2 = 1A = B$; „ w_2.
(iv) Three washes $= 3 = 2A = 1B = C$; „ w_3.
(v) Four washes $= 4 = 3A = 2B = 1C = D$; „ w_4.
(vi) Five washes $= 4A = 3B = 2C = 1D$; „ w_5.
(vii) Six washes $= 4B = 3C = 2D$; „ w_6.
(viii) Seven washes $= 4C = 3D$; „ w_7.
(ix) Eight washes $= 4D$; „ w_8.

This scale of nine values is probably the best we can make to correspond with the density of colour in the 1914 observations, i.e. we may put as equivalents

1914	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*+*I*+*K*
1920	w_8	w_7	w_6	w_5	w_4	w_3	w_2	w_1+w_0

Seven eggs were returned as blue and two were by oversight not specified at all with regard to ground-colour; this reduced the number of eggs available for colour value reduction from 956 to 947.

(ζ) Types of Nest; these were classified into (n_0) Nothing at all; (n_1) A few bits of odds and ends; (n_2) An attempt at nest making; (n_3) Moderately made; (n_4) Well made.

Plate III gives a few specimens of these categories.

(θ) Environment of Nest. In 1920 the environment of the nests was classified in the following manner: (e_1) Green plants; (e_2) Grey-blue, mostly pebbles and flints; (e_3) Grey, mostly with sand; (e_4) Speckled shingle; (e_5) Brown sand; (e_6) Sand and psamma, dull brown.

This classification is as far as possible according to colour; its principal use will be to examine whether there is any relation between the colour and mottling of the egg and its environment. This might be expected, if the colouring were protective.

It was pointed out in the previous paper that we ought—if the egg be a solid of revolution—to have:

$$\pi = \frac{\text{Mean Transverse Girth}}{\text{Mean Breadth}} \times (1 - r_{GbB}\, v_{Gb} v_B + v_B^2),$$

the factor in brackets, however, being so close to unity that for our purposes it may be replaced by unity. We find on the present occasion $\pi = 3{\cdot}1536$ from this formula as against $\pi = 3{\cdot}2071$ for 1913 and $3{\cdot}2237$ for 1914. This is a very considerable improvement, and is probably due to the specially constructed celluloid tapes being replaced by strips of paper, which were afterwards measured on a scale. This test is a useful control on the accuracy of the fieldworkers' measurements.

(4) *Influence of Breadth of Egg on Mottling and Ground-Colour.* We have again considered the possible effect of pressure on the egg as it passes through the oviduct in influencing the amount of pigment deposited. As before we can do this in two ways, namely by considering (i) whether in the same clutch the broader egg has more pigmentation and (ii) whether in the population in general there is an association between breadth and pigmentation. It is clear that we may consider the problem of colour from the three standpoints (*a*) intensity of green, (*b*) intensity of brown, or (*c*) the combined intensity (or number of washes whether green or brown). In dealing with mottling we may either measure (*d*) the size of the blotches—in which case our scale of mottling indicates that *a*, *b*, *c* and *d* may be grouped together as fine, *e*, *f*, *g* and *h* as coarse, *i* as markedly coarse and *j*, *k* as confluent. Or, we may measure (*e*) the density of the belt, disregarding whether it is equatorial or polar. From this standpoint: *a*, *e*, *i* have no belt, they are followed in order of belt intensity by *b*, $c + d$, *f*, $g + h$, $j + k$. We have accordingly to judge the broader egg of each pair from the same clutch from these five standpoints.

The broader egg in every possible clutch pair has:

Larger blotches in 121 cases.	Greater density of belt in 183 cases.
Equal sized blotches in 464 cases.	Equal density of belt in 266 cases.
Smaller blotches in 137 cases.	Smaller density of belt in 273 cases.
Greater intensity of green in 116 cases.	Greater intensity of brown in 213 cases.
Equal intensity of green in 482 cases.	Equal intensity of brown in 322 cases.
Smaller intensity of green in 121 cases.	Smaller intensity of brown in 184 cases.

Greater combined intensity in 243 cases.
Equal combined intensity in 283 cases.
Smaller combined intensity in 193 cases*.

* In the case of the mottling 44 pairs had to be omitted; 38 of these had the same breadth of egg in both members of the pair; while in the remaining six, the mottling was not specified for one or other of the eggs. Similarly in the case of ground-colour, 38 pairs had the same breadth; in seven pairs one or other of the eggs was blue, and in the remaining two pairs the colour of one or other of the eggs was not specified. Thus all our 766 pairs are in both cases accounted for.

If the pigmentation conditions were not associated with the breadth of the egg, we should expect the greater and smaller intensities to be equally divided. Thus in 258 unequal cases we should expect 129 with larger and an equal number with smaller blotches; the standard deviation is $\sqrt{\frac{1}{2}\frac{1}{2}258} = 8{\cdot}03$, and accordingly the observed result is represented by a deviation of $8 \pm 5{\cdot}42$, which is not significant. We cannot therefore assert that *size* of blotches is influenced by the breadth of the egg.

On the other hand if we take the density of the belt, there was a smaller density in 273 out of 456 cases or a deviation of $45 \pm 6{\cdot}79$. This is markedly significant, or a broad egg has a small density of belt. The result may be taken to be in accordance with that obtained from the 1914 data*, where the scale of mottling was different, and an attempt was made to measure by eye the *total amount* of mottling; there is little doubt that this would be much influenced by the density of the belt. We must, we think, conclude that the glands which distribute the mottling pigment do so *less freely* when the eggs are larger, i.e. exert more pressure on the walls of the oviduct.

We now turn to the ground-colour, and find that for green the deviation from equality is only $2{\cdot}5 \pm 5{\cdot}19$. The breadth of the egg has therefore no relation to the green pigment. On the other hand in the case of brown pigment there is a deviation of $14{\cdot}5 \pm 6{\cdot}72$, or the odds are roughly of the order of about 13 to 1 against such an excess. It seemed well to test this further and we took a diagonal grouping of pigment intensity on our colour scale. This gave a deviation of

TABLE I.

Density of Belt†.

Breadth of Egg‡	None $a+e+i$	Slight b	Moderate $c+d$	Medium f	Dense $g+h$	Very Dense $j+k$	Totals
2·65—	—	—	—	1	—	—	1
2·70—	2	—	—	—	1	—	3
2·75—	1	—	—	—	1	—	2
2·80—	2	1	1	—	1	1	6
2·85—	7·5	2	3	1·5	1	1	16
2·90—	16·5	10·5	20	5	1	4	57
2·95—	43·5	17	21·5	15·5	13·5	3	114
3·00—	97	31	44·5	23·5	22·5	8·5	227
3·05—	89·5	23	36·5	21	18	4	192
3·10—	103	26·5	25·5	32·5	18·5	6	212
3·15—	38	7·5	8	14	7·5	1	76
3·20—	20·5	2·5	4·5	3	3	0·5	34
3·25—	4·5	1·5	2	1	—	1	10
Totals	425	122·5	166·5	118	88	30	950
Means	3·0659	3·0478	3·0395	3·0658	3·0518	3·0333	3·0566

* See *Biometrika*, Vol. XII. p. 310.

† Six eggs had to be excluded owing to absence of record.

‡ 3·05— indicates that the eggs of breadths between 3·045 and 3·095 were included in the group, the eggs being recorded to two decimal places. The centre of the group 3·05— is accordingly 3·07.

$25 \pm 7{\cdot}04$ from equality, or the odds are about 104 to 1 against such an excess of total pigmentation occurring with the broader egg on the mere basis of chance occurrence. The fact of the broader egg having more ground pigment is therefore probably significant.

It seems desirable to consider these results from the standpoint not of the individual bird, but from that of the whole population. Are broader eggs in general associated with less density of belt, and with greater density of ground-colour? Tables I and II provide the requisite data to answer this.

TABLE II.

*Value of Ground-Colour**.

Breadth of Egg	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	Totals
2·65—	—	—	—	—	—	1	—	—	—	1
2·70—	—	—	—	—	—	2	1	—	—	3
2·75—	—	—	—	—	1	—	—	—	1	2
2·80—	—	—	—	—	1	1·5	2·5	1	—	6
2·85—	—	—	—	—	2	6	3	5	—	16
2·90—	—	—	1	5	15·5	11	11	13·5	1	58
2·95—	—	—	2	6	18	27	26·5	26·5	8	114
3·00—	—	—	1·5	16	44	64	50	39	11·5	226
3·05—	—	—	3	16·5	36·5	48·5	40·5	35	13	193
3·10—	—	—	1	22	40·5	50·5	44	36	16	210
3·15—	—	—	2	7	15	16	13	19	3	75
3·20—	—	—	1	3	1	9	9	7	3	33
3·25—	—	—	—	1	—	4	1	4	—	10
Totals	—	—	11·5	76·5	174·5	240·5	201·5	186	56·5	947
Means	—	—	3·0678	3·0739	3·0494	3·0536	3·0535	3·0575	3·0651	3·0561

It is obvious that in the first table the means are very irregular, and there is little hope of a smooth regression curve. We have in fact:

$$\eta^2_{y.x} = {\cdot}006{,}124, \text{ while } \bar{\eta}^2_{y.x} = {\cdot}007{,}391;$$

or we cannot say that there is any significant association between breadth of egg and density of belt.

If we pass to the second table we find:

$$\eta^2 = {\cdot}018{,}147, \text{ while } \bar{\eta}^2 = {\cdot}005{,}263.$$

Hence η appears significant, and without correction its value is ·1347. Indeed if we take the mean resulting from clubbing the frequencies w_3 and w_4 together, namely 3·0569, we find the regression curve could be very adequately given by a parabola. This, however, suggests that the correlation coefficient will be very nearly zero. The results certainly do not suggest a considerable *positive* correlation between intensity of pigment and breadth of egg. At first sight the results for the individual and the population appear to be contradictory; but on further investigation they will be found to be possibly rather suggestive. For when we

* Nine eggs had to be excluded owing to no record of ground-colour.

confine our attention to a particular egg-layer we must be dealing with a *partial* correlation. Let H denote the sum of the factors in the hen which determine the nature of its egg, B be breadth of egg, z be density of mottling in the belt or zone, w = intensity of ground-colour. Then when we deal with the eggs of the same clutch we are considering

$$r_{zB.H} \text{ and } r_{wB.H}$$

or the expressions

$$r_{zB.H} = \frac{r_{zB} - r_{zH}\, r_{BH}}{\sqrt{1 - r^2_{zH}}\sqrt{1 - r^2_{BH}}}, \quad r_{wB.H} = \frac{r_{wB} - r_{wH}\, r_{BH}}{\sqrt{1 - r^2_{wH}}\sqrt{1 - r^2_{BH}}}.$$

Now if as we have seen reason to believe the correlation coefficients r_{zB} and r_{wB} in the population at large are very small or even insignificant, it must follow that $r_{zB.H}$ will be *negative*, if both r_{zH} and r_{BH} are positive, i.e. if the factors (including dimension of oviduct) which tend to make a broad egg also tend to make a more densely pigmented zone or belt. Again $r_{wB.H}$ will be positive if the factors which tend to make an egg broad also tend to give it less intensity of ground-colour (r_{wH} *negative*). Suppose we consider for a moment the oviduct O to be an important part of H. Now certainly we should anticipate that r_{BO} is a considerable positive correlation. We should then have to suppose that a broad oviduct marked a dense belt, but a pale ground-colour in the egg. Neither of these suppositions appears unreasonable. In fact it was an overlooking of the difference between partial and total correlation, when r_{zB} and r_{wB} are negligible, which has led us to difficulty in the interpretation. The correctness of the suppositions can only be determined when we have more knowledge of the factors that determine both mottling and ground-colour.

(5) *Change of Type of Egg with Season.*

(i) Comparison of the absolute measurements and indices for the three seasons.

TABLE III.

Character	Mean		
	Season 1913	Season 1914	Season 1920
Length L (cms.)	4·14 ± ·007	4·21 ± ·004	4·19 ± ·004
Breadth B (cms.)	2·98 ± ·004	3·01 ± ·002	3·06 ± ·002
Longitudinal Girth G_l (cms.) ...	11·39 ± ·015	11·56 ± ·007	11·37 ± ·007
Transverse Girth G_b (cms.) ...	9·59 ± ·014	9·66 ± ·006	9·65 ± ·006
Index 100 B/L	72·04 ± ·136	71·75 ± ·070	73·12 ± ·068
Index of Ovality O*	56·35 ± ·171	55·81 ± ·088	57·72 ± ·083
Bulk LB^2 (cm.^{3})	36·69 ± ·122	38·26 ± ·080	39·11 ± ·065

This table indicates that the eggs of 1920 were significantly greater in bulk as measured by LB^2. This is due to the significant increase in breadth, the length

* We have taken $O = \frac{1}{2}B/(L - \frac{1}{2}B)$, so that a small value of the index corresponds to a large value of the ovaloid character.

and the longitudinal girth having decreased sensibly. This girth is indeed below that of 1913. Both indices—100 B/L and the index of ovality—have increased, but this is what we should anticipate when the breadth increases and the length decreases. While the eggs of the 1914 season were in length, breadth and the two girths significantly larger than those of 1913*, there has however been a continuous change in bulk owing to the continuous increase in breadth. Unfortunately the war years give a gap of six years, and it would be hazardous to assert that we have here a secular change in shape. What appears to be secular may after all only be seasonal variation, when we have only three years for comparison.

(ii) Comparison of the variabilities in the three seasons.

We have the following table:

TABLE IV.

Character	Standard Deviation			Coefficient of Variation		
	1913	1914	1920	1913	1914	1920
Length L	·180 ± ·035	·185 ± ·003	·169 ± ·002	4·34 ± ·12	4·39 ± ·09†	4·04 ± ·09
Breadth B	·099 ± ·010	·099 ± ·001	·085 ± ·001	3·33 ± ·09	3·28 ± ·07	2·77 ± ·06
Longitudinal Girth G_l	·376 ± ·010	·350 ± ·005	·333 ± ·005	3·30 ± ·09	3·03 ± ·06	2·93 ± ·07
Transverse Girth G_b	·347 ± ·010	·300 ± ·004	·264 ± ·004	3·62 ± ·10	3·10 ± ·07	2·74 ± ·06
Index 100 B/L ...	3·449 ± ·096	3·479 ± ·050	3·057 ± ·048	4·79 ± ·13	[4·84 ± ·07]‡	[4·18 ± ·10]‡
Index of Ovality O ...	4·334 ± ·121	4·326 ± ·062	3·778 ± ·059	7·69 ± ·22	[7·75 ± ·11]	[6·55 ± ·10]
Bulk LB^2 (cm.^{3}) ...	3·095 ± ·086	3·294 ± ·047	2·936 ± ·046	8·44 ± ·35	8·61 ± ·18	7·48 ± ·17

Taken all round—whether we consider standard deviations or coefficients of variation—this table shows a marked decrease in variability between 1920 and the two earlier seasons. The tendency to decreased variability was not all in one direction in 1914 as compared with 1913, and the changes themselves were nothing like so large. In the absence of information for the years 1915—1919, it is not possible to say this corresponds to a secular change; such a secular change does not exist for 1913 to 1914. Again, the absence of records of the season 1919 gives us no clue to the fate of the colony in that year, e.g. whether it was a year of stringent selection, which left only the more mature birds to breed in 1920. The remarks on p. 296 at any rate suggest that the fieldworkers may in 1920 have dealt largely or at least more largely than those of 1913 and 1914 did with second nesters and so have missed the first eggs of birds which are often smaller ones. It is clear that in another year it would for this reason be worth while starting the egg measurements as soon as the birds begin to lay. If the smaller variability is not to be accounted for by the mere fact that the measurements

* See footnote p. 308.

† This and the succeeding three probable errors were by a slip given erroneous values in the last paper.

‡ See remarks in footnote † on p. 147 of first memoir, *Biometrika*, Vol. x.

were made at a particular time in the nesting season, then it is clear that 1920 was markedly different from 1913* or 1914, and that very considerable changes in variability can occur from year to year. The interesting point, namely as to whether this reduction of variability is merely seasonal, or is a permanent or secular change, can only be ascertained when another season at least has been dealt with.

(6) *Association of Type of Nest and its Environment with Egg Characters.*

(i) We will deal first with the size and shape of the egg. Tables V and VI give the results in two forms, first, the mean values for each type of nest, and secondly the correlations. In the first table the differences between the means of all types and the means for the arrays are not always markedly significant having regard to the probable errors of the differences. Thus in the case of Longitudinal Girth the difference in girth between the eggs in quite unmade nests and those in moderately made nests is ·075 ± ·032, which cannot be pronounced as definitely significant, while that between eggs in moderately and in well-made nests is ·095 ± ·024, and is significant. Again, the distinction for breadth between eggs in nests with no making and nests with a few bits is ·021 ± ·008 and is again non-significant, while the difference in breadth between eggs in nests with some attempt at making and those in well-made nests is ·036 ± ·006, and is definitely significant. To have tabled a number of probable errors of this kind would hardly have done more than convince the reader that the type of nest is in some respects related to the size of the egg, but that the relation was of a not easily unriddled type. Accordingly it seemed best to examine the table as a whole and see if any general facts could be drawn from it. It appears fairly clear on inspection that the large eggs are either in badly built or in well-built nests, and that the same type of nests have the rounder eggs or the eggs of a less ovaloid character. This relationship is not a very marked one, but it certainly appears to be significant. The regression lines therefore of size or shape of egg on degree of care in building

TABLE V.

Mean Shape and Size Characters, corresponding to Nest Types.

Nest Type	Means					
	Length	Longitudinal Girth	Breadth	Bulk (LB^2)	Index	Index of Ovality
All types	4·185	11·374	3·057	39·108	73·110	57·700
n_0 = Nothing at all ...	4·190	11·388	3·071	39·450	73·368	57·965
n_1 = A few bits	4·202	11·398	3·050	39·105	72·692	57·207
n_2 = An attempt at making	4·191	11·378	3·046	38·992	72·801	57·281
n_3 = Moderately made ...	4·159	11·313	3·052	38·722	73·510	58·187
n_4 = Well made	4·185	11·408	3·082	39·725	73·674	58·476

* The reader must remember that 1913 was a bad season and 1914 a good one with plenty of the whitebait which failed in 1913: see *Biometrika*, Vol. XII. p. 311.

nest are not straight lines but tend to be parabolic in character. We have accordingly found the correlation ratio in these cases, compared it with the mean $\bar{\eta}$ for

TABLE VI.

Correlations of Nest Type with Shape and Size.

Character Pair	1920		1914	1920
	η	$\bar{\eta}$	Biserial r	Biserial r
Type of Nest and L	·0906	·0648 ± ·0219	+ ·0953 ± ·0321	− ·0890 ± ·0277
" " G_1	·1053		+ ·0685 ± ·0322	− ·0622 ± ·0278
" " B	·1568		− ·0952 ± ·0321	+ ·1040 ± ·0277
" " LB^2	·1133		− ·0223 ± ·0322	+ ·0128 ± ·0277
" " I	·1364		− ·1372 ± ·0319	+ ·1526 ± ·0274
" " O	·1426		− ·1492 ± ·0318*	+ ·1602 ± ·0273

no association to determine its significance, and corrected it in all cases for the class index correction. The results are given in Table VI. This table enables us to say that there is definite, if not very marked association between both indices and the Type of Nest; there is a slightly more emphasised association between Breadth and the Type of Nest, and *possibly* an association between Bulk and Longitudinal Girth and Type of Nest, but not a definitely significant one between Length and Type of Nest, unless we look at the general resemblance of the length column to the later columns in Table V. It is well known that relatively small associations found in different ways may lead to somewhat divergent results. While the correlation coefficient is always less than the correlation ratio if the product moment method and the same arrays be used for both, this by no means follows universally if the correlation coefficient be found by other processes, such as the tetrachoric process, or the "biserial r" method. To check to some extent the above results, r was obtained by the biserial method, dividing between groups n_2 and n_3, i.e. separating off the moderately made and well made groups. Thus there resulted values comparable with those of 1914, which are given in the preceding column. It will be clear that the biserial correlation coefficients of Type of Nest with Length and Longitudinal Girth are of opposite sign in the two years and this tends to confirm the conclusion already based on the correlation ratios that these associations are of doubtful significance. Both years agree in giving the association of Bulk of egg with Type of Nest as insignificant. The disagreement in the sign of the Breadth correlation is, however, troublesome as both appear significant. It must, however, be noticed that the signs of the Breadth/Length Index and the Index of Ovality agree with the Breadth sign as they should do, so that all three are opposite in 1914 to what they are in 1920. We think this change of sign is probably explicable on the ground of the parabolic regression. It is difficult to explain how the birds laying narrow and oval eggs in one year could make the best nests,

* The inverse of our present index is used on p. 315 of the second paper and accordingly for comparison the sign there given must be changed.

while six years later the birds laying broad and round eggs should make the best nests. But when we fit a straight line to a parabolic curve, a very slight change in the frequency may tilt either end of the line up or down. Again the record for 1914 was for only two types of nest, simple or unelaborated and complex or elaborated. The percentage of simple nests, i.e. 12·9, does not unfortunately coincide with any division we can make on our present material; it would cover all the cases of no nest and about one-quarter of those now described as made of odds and ends. While we are compelled, if we pay attention to the probable errors, to say that on the present data there is some association between nest type and breadth and rotundity of egg, we have to admit: (*a*) that it is not of any marked intensity, (*b*) that the exact nature of it is not clear, i.e. why birds laying broader and rounder eggs should have the worst and the best nests, and (*c*) that we cannot in view of the results for 1914 assert at present a secular relation of this kind. We have not succeeded in finding any really important association between the size and shape of the egg and the type of nest.

We now turn to the same egg characters correlated with the environment of the nest. Here we notice in Table VII that the largest eggs—whether as to length, longitudinal girth, breadth or bulk—are laid with a grey-blue background, which consists mostly of pebbles and flints. The category with the next largest eggs is clearly the brown sand.

TABLE VII.

Mean Shape and Size Characters corresponding to Environment of Nest.

Environment of Nest	Means					
	Length	Longitudinal Girth	Breadth	Bulk	Index	Index of Ovality
e_1 = Green Plants ...	4·193	11·374	3·046	38·883	72·655	57·183
e_2 = Grey-blue, mostly pebbles and flints	4·223	11·484	3·080	40·033	73·019	57·551
e_3 = Grey, mostly with sand	4·178	11·347	3·035	38·524	72·776	57·385
e_4 = Speckled shingle .	4·167	11·332	3·055	38·944	73·405	58·028
e_5 = Brown sand ...	4·190	11·386	3·062	39·269	73·184	57·768
e_6 = Sand and psamma, dull brown	4·150	11·288	3·046	38·595	73·500	58·250

The smallest eggs taken all round are to be found where the psamma mingles with the sand. The most dolichoöidal of the eggs are to be found with a background of green plants, while the most brachyoïdal favour regions where the psamma grows or the shingle is speckled. It is not possible, however, to assert definitely that the indices are related to the environment of the nest significantly; this appears from an examination of Table VIII, where it will be seen that the η for the indices is not significantly different from the mean correlation ratio $\bar{\eta}$ for no association. The same table indicates that the relations of Breadth, Bulk and possibly Length with Environment of Nest, if not very noteworthy, are significant.

TABLE VIII.

Correlations of Environment and Shape and Size of Egg.

Character Pair	η	$\bar{\eta}$
Environment and L ...	·1289	
" " G_1 ...	·0570	
" " B ...	·1687	·0728 ± ·0219
" " LB^2 ...	·1696	
" " I ...	·0981	
" " O ...	·0893	

It may well be that the more exposed situations, such as pebbles and brown sand, are sought by the stronger and older birds, and that these birds lay larger eggs*. It does not follow because there is some association that the source of it is protection. It is noteworthy that the type of environment appears as closely related to the size of the eggs as the type of nest—indeed it is possibly more closely related.

(ii) Such a result suggests that it is important to ascertain the correlation between our scale of Environments and the Type of Nest. We have 405 nests for which environment is specified and we can either correlate nest and environment with or without weighting with the number of eggs therein. We proceeded by the mean square contingency method correcting for class index correlations. There resulted:

$$C_2 = \cdot3792, \quad \bar{C}_2 = \cdot1729 \pm \cdot0150,$$

for nests weighted with eggs; and

$$C_2 = \cdot3666, \quad \bar{C}_2 = \cdot2585 \pm \cdot0217$$

without weighting.

Both of these are definitely significant.

An examination of the signs of the contingencies of the individual cells of the tables shows that although they are somewhat irregularly distributed it is the green-grey environment rather than the brown which is associated with the more elaborated nests. To test this fourfold tables were made dividing between e_3 and e_4 of the environment categories, first between n_2 and n_3 and then between n_1 and n_2 of the nest categories. In the first case tetrachoric r was positive and less than its probable error; in the second case r was $= -\cdot16$ and significant. It is clear therefore that the relatively high association exhibited by the mean square contingency coefficient is not wholly due to the poor nests being on the brown environments. In the green environment (e_1) there is the probable number of well made nests, a large excess of the moderately made nests, and defect of all the lower categories of nests. On the grey-blue environment (e_2), mostly pebbles and flints, there is a marked excess of attempts at making and a corresponding defect

* It does not of course follow that a stronger and larger bird will necessarily lay a large egg, e.g. the Cuckoo.

of moderately made nests, while well-made nests are somewhat above the random frequency. Among the grey, mostly sandy environment (e_3), the nests are chiefly attempted makings or lacking altogether; the other categories are defective. In the speckled shingle (e_4), there is a large excess of bits and chips. The brown sand (e_5) seems to lead to the birds making no nests at all, or well-made nests. Finally the sand and psamma, a dull brown (e_6), has an excess of the mediocre nests.

Looked at from the other aspect the excess of clutches without nests (n_0) is to be found in the brown sand and sand and psamma; the excess of nests involving chips and bits (n_1) is to be found in the speckled shingle; the excess of attempted nests (n_2) in the grey-blues and greys, i.e. among the pebbles, flints and mixtures of these with sand; the moderately made nests (n_3) are in excess in the green environment and to some extent in the brown environment, i.e. at the colour extremes; while lastly the excess of the best made nests (n_4) occurs chiefly in the brown sand. It must be admitted that in some manner the nest is finished to its environment, but it would need a more detailed study than has yet been made to come nearer to the clues. Moisture of surface, lighting, hardness of substratum would probably have to be taken into account as well as colour and ease of access to material. The latter is possibly not of much importance to a bird, whose flight

TABLE IX.

Type of Nest and Type of Environment.

Type of Nest.

Type of Environment		Nothing at all (n_0)	A few bits and chips (n_1)	Some attempt at making (n_2)	Moderately made (n_3)	Well made (n_4)	Totals
Green Plants (e_1)	Observed	1	25	32	50	21	129
	Random	9·1	29·0	42·0	28·2	20·7	
	Excess	− 8·1	− 4·0	−10·0	+21·8	+ 0·3	
Grey-blue, mostly pebbles and flints (e_2)	Observed	11	32	74	20	36	173
	Random	12·2	38·9	56·3	37·8	27·8	
	Excess	− 1·2	− 6·9	+17·7	−17·8	+ 8·2	
Grey, mostly with sand (e_3)	Observed	16	22	60	23	14	135
	Random	9·5	30·4	43·9	29·5	21·7	
	Excess	+ 6·5	− 8·4	+16·1	− 6·5	− 7·7	
Speckled shingle (e_4)	Observed	12	95	110	73	44	334
	Random	23·6	75·1	108·7	73·0	53·6	
	Excess	−11·6	+19·9	+ 1·3	− ·0	− 9·6	
Brown sand (e_5)	Observed	19	32	23	25	37	136
	Random	9·6	30·6	44·2	29·8	21·8	
	Excess	+ 9·4	+ 1·4	−21·2	− 4·8	+15·2	
Sand and psamma, dull brown (e_6)	Observed	8	7	9	16	0	40
	Random	2·8	9·0	13·0	8·7	6·4	
	Excess	+ 5·2	− 2·0	− 4·0	+ 7·3	− 6·4	
Totals	—	67	213	308	207	152	947

of a couple of seconds would carry it to a nesting ground and material such as it might desire*. All we can say at present is that the type of nest does appear to be quite significantly related to the environment, but that the nature of the relation is not very clear upon the present categories. We give above (Table IX) the contingency table on which the above remarks are founded, using the one in which the nests are weighted with the number of the clutch.

(iii) Pigmentation of Egg in relation to the Type of Nest and the Environment of Nest (Appendix, Table M). Turning first to mottling we may consider Density of Belt in the same group as Table I, or we may consider merely the size of blotches, grouping a, b, c and d together as fine markings; e, f, g and h together as coarse markings and i, j and k as confluent blotches. The two tables are:

TABLE X.

Density of Belt and Type of Nest.

Density of Belt.

Type of Nest	$a+e+i$	b	$c+d$	f	$g+h$	$j+k$	Totals
n_0	21	10·5	15	11	6	3·5	67
n_1	93	28	40	25·5	23·5	8	218
n_2	129	41	47·5	46·5	35	7	306
n_3	98·5	20	40·5	23	12·5	10·5	205
n_4	81·5	22	22·5	12	11	1	150
Totals	423	121·5	165·5	118	88	30	946

TABLE XI.

Coarseness of Mottling and Type of Nest.

Coarseness of Mottling.

Type of Nest	$a+b+c+d$	$e+f+g+h$	$i+j+k$	Totals
n_0	35	26·5	5·5	67
n_1	121·5	74·5	22	218
n_2	155	131·5	19·5	306
n_3	125·5	64·5	15	205
n_4	81·5	62	6·5	150
Totals	518·5	359	68·5	946

These tables were analysed by mean square contingency† and it was found that:

For Type of Nest and Density of Belt:

$$C_2 = \cdot 1747, \quad \bar{C}_2 = \cdot 1725 \pm \cdot 0151.$$

* If site were more influential than type of nest and the bird naturally lazy, then type of nest might depend on material of the immediate environment. Professor Oliver rather inclines to this view.

† Uncorrected for class index corrections. Actually in the first case

$$\phi^2 = \cdot 031{,}489 \text{ and } \bar{\phi}^2 = \cdot 030{,}655 \pm \cdot 0055;$$

and in the second $\phi^2 = \cdot 013{,}584$ and $\bar{\phi}^2 = \cdot 014{,}799 \pm \cdot 0039$.

For Type of Nest and Coarseness of Mottling:

$$C_2 = \cdot1158, \quad \bar{C}_2 = \cdot1208 \pm \cdot0221.$$

Thus C_2 is insignificantly different from $\bar{C}_2$, the average coefficient of mean square contingency for zero association.

We can accordingly quite definitely assert that there is no significant relation in the present data to be found between Type of Nest and the Mottling of the egg, whether the latter be judged by the density of the belt or the coarseness of the mottling.

We now turn to the problem of whether the Ground-Colour is related to the Type of Nest. We may consider the amount of green or the amount of brown ground-colour (see Appendix, Table O). We find for contingency *:

Amount of Green and Type of Nest:

$$C_2 = \cdot1474, \quad \bar{C}_2 = \cdot1404 \pm \cdot0154.$$

Amount of Brown and Type of Nest:

$$C_2 = \cdot1731, \quad \bar{C}_2 = \cdot1577 \pm \cdot0154.$$

In neither case is there a significant difference between C_2 and $\bar{C}_2$. We may try the same problem in a third way by density of ground-colour pigmentation, the categories being the w's of our p. 302. We have the following table:

TABLE XII.

Density of Ground-Colour and Type of Nest.

Type of Nest	Density of Ground-Colour.							
	w_2	w_3	w_4	w_5	w_6	w_7	w_8	Totals
n_0	2	4	12	26	8	13	2	67
n_1	1	17	42·5	58	42·5	45	11	217
n_2	5·5	29·5	54·5	71·5	69	59	17	306
n_3	1	14	31·5	51·5	52	38·5	14·5	203
n_4	2	11	33	32·5	29	30·5	12	150
Totals	11·5	75·5	173·5	239·5	200·5	186	56·5	943

The uncorrected contingency is †:

$$C_2 = \cdot1543 \quad \text{and} \quad \bar{C}_2 = \cdot1865 \pm \cdot0149.$$

Hence the result is again insignificant, or we must conclude that the present data show no evidence whatever of either Mottling or Ground-Colour (for shade or value) being associated with the Type of Nest. The nest does not seem built in any association with the colouring of the eggs.

* We have $\phi^2 = \cdot022{,}229$ and $\bar{\phi}^2 = \cdot020{,}148 \pm \cdot0045$ in the first case,
and $\phi^2 = \cdot030{,}898$ and $\bar{\phi}^2 = \cdot025{,}451 \pm \cdot0051$ in the second;
so that, as in the previous case, it is idle to correct C_2.

† We have $\phi^2 = \cdot024{,}384$ and $\bar{\phi}^2 = \cdot036{,}055 \pm \cdot0060$.

We next turn to the question of environment and we ask in the same way: Is it possible that there is any relation between the nature of the environment and either the Mottling or Ground-Colour of the eggs laid thereby? Just as before, we can form two tables for Mottling: one for density of belt and one for coarseness of mottling. We have:

TABLE XIII.

Density of Belt and Type of Environment.

Density of Belt.

Type of Environment	$a+e+i$	b	$c+d$	f	$g+h$	$j+k$	Totals
e_1	51·5	13	26·5	22·5	12·5	2	128
e_2	74·5	18	26	25·5	21·5	7·5	173
e_3	54·5	17	21	15·5	20·5	7·5	136
e_4	152·5	53	68·5	30	20·5	8·5	333
e_5	70·5	17·5	16·5	19·5	9	2	135
e_6	18·5	4	8	4	4	1·5	40
Totals	422	122·5	166·5	117	88	29	945

TABLE XIV.

Coarseness of Mottling and Type of Environment.

Coarseness of Mottling.

Type of Environment	$a+b+c+d$	$e+f+g+h$	$i+j+k$	Totals
e_1	69	49	10	128
e_2	75·5	82·5	15	173
e_3	70	53	13	136
e_4	201·5	110	21·5	333
e_5	78·5	50	6·5	135
e_6	25·5	13	1·5	40
Totals	520	357·5	67·5	945

We find*: for Density of Belt and Type of Environment:

$$C_2 = \cdot 1965, \quad \bar{C}_2 = \cdot 1890 \pm \cdot 0149;$$

and for Coarseness of Mottling and Type of Environment:

$$C_2 = \cdot 1490, \quad \bar{C}_2 = \cdot 1330 \pm \cdot 0156.$$

It is clear that in neither case whether we judge by the C_2's or the ϕ^2's can we assert that the results are significant. Or, on such numbers as are available we cannot assert any association between the Environment of the Nest and the Mottling of the eggs. If larger numbers reduced the probable error so that we could assert significance, it is clear that we should still have to admit an absence of any *marked* association.

* For the first case $\phi^2 = \cdot 040{,}169$ and $\bar{\phi}^2 = \cdot 037{,}037 \pm \cdot 0061$; and for the second $\phi^2 = \cdot 022{,}741$ and $\bar{\phi}^2 = \cdot 017{,}989 \pm \cdot 0043$. The C_2's are uncorrected values, which must be used to compare C_2 and $\bar{C}_2$.

We now turn to the relation between Ground-Colour and Type of Environment, we find:

For Type of Environment and Amount of Green in Ground-Colour*:

$$C_2 = \cdot1867, \quad \bar{C}_2 = \cdot1543 \pm \cdot0156.$$

For Type of Environment and Amount of Brown in Ground-Colour:

$$C_2 = \cdot1697, \quad \bar{C}_2 = \cdot1729 \pm \cdot0154.$$

The latter value is clearly insignificant, or if there be any relation between brown of the ground-colour of the egg and the environment, it is not sufficiently intense to be shown in a series of 942 eggs of the Common Tern. The former value is such that C_2 differs from $\bar{C}_2$ by slightly more than double its probable error, or 1·58 times the S.D.; such a deviation would be likely to occur once in eighteen trials and some would consider such a deviation possibly significant. Assuming it to be so, and correcting for both class indices, we find:

$$C_2 = \cdot2132.$$

Even if we suppose the significance to be possible and take this corrected value, it is a small association and not evidence of any marked protective colouring. To determine its nature we have formed a fourfold table as follows:

Ground-Colour of Egg.

Tone of Environment.	Light Green 1+2	Dark Green 3+4	Totals
Green Greys $e_1+e_2+e_3$	113	318	431
Browns $e_4+e_5+e_6$ ...	105·5	405·5	511
Totals	218·5	723·5	942

This leads us to $r = \cdot114 \pm \cdot0378$ and indicates that the contingency relation does not represent a clear cut relation, but is largely a product of random sampling. As a matter of fact one-third of the mean square contingency is contributed by an excess of seven light green eggs among the grey-blue pebbles and flints. We fear it is impossible on such evidence to assert that the ground-colour of the terns' eggs is matched by the tone of the environment.

Must the reader therefore conclude that egg pigmentation—at any rate among the terns—is not now protective? This does not seem to us to follow at all. We have been considering whether *within the species*, i.e. intra-racially, there is any protective colouring in relation to nest or its environment, and we have not been able to discover that the variations within the species are essentially associated with make or site of nest. It is quite possible that relative to other species' egg coloration the

* The C_2's are uncorrected. In the first table we have 1, 2, 3, 4 for categories of green, and in the second 0, *A*, *B*, *C*, *D* for categories of brown, or the first table has 24 and the second 30 cells, which accounts for the different values of C_2. We have in the first case:

$$\phi^2 = \cdot036{,}125, \ \bar{\phi}^2 = \cdot024{,}416 \pm \cdot0050, \text{ which might possibly be significant,}$$

and in the second case

$$\phi^2 = \cdot029{,}638, \ \bar{\phi}^2 = 030{,}785 \pm \cdot0055, \text{ which is certainly not significant.}$$

tern's may have high association with the environment, i.e. there may be inter-racial association of pigmentation and locus; the grebe or the petrel's eggs might be very conspicuous with the same background as the tern's. We cannot assert therefore that the pigmentation of the tern's egg is non-protective. At the same time the tern's egg has a wide range of ground-colour and much variability of mottling, and it would have been some satisfaction for believers in the protective pigmentation of the egg had we been able to demonstrate that even intra-racially, when there exists much variation in the colouring, some definite association exists between that colouring and its environment. All we can say is that on our data this has not been possible.

(7) *The Organic Correlations in Size and Shape.*

The following table gives the results for the three seasons.

TABLE XV.

Organic Correlations in the Three Seasons.

Character Pair	Symbols	Correlation		
		1913	1914	1920
Length and Breadth	L, B	·2220 ± ·0374	·2104 ± ·0193	·2748 ± ·0202
Longitudinal and Transverse Girths	G_l, G_b	5297 ± ·0284	·5139 ± ·0149	·5560 ± ·0152
Length and Longitudinal Girth ...	L, G_l	·8804 ± ·0088	·8515 ± ·0055	·8895 ± ·0048
Breadth and Longitudinal Girth ...	B, G_l	·5216 ± ·0286	·4840 ± ·0155	·5481 ± ·0153
Index and Length	I, L	− ·7284 ± ·0185	− ·7577 ± ·0086	− ·7818 ± ·0085
Index and Breadth	I, B	·5033 ± ·0294	·4537 ± ·0161	·3981 ± ·0184
Index and Longitudinal Girth ...	I, G_l	− ·3832 ± ·0336	− ·4496 ± ·0161	− ·4804 ± ·0168

In order to determine which differences are significant, the following table of differences and their probable errors was constructed.

TABLE XVI.

Seasonal Changes in Correlation.

Character Pair	$\Delta = 1914 - 1913$	$\Delta' = 1920 - 1913$	$\Delta'' = 1920 - 1914$
L and B	− ·0116 ± ·0421	·0528 ± ·0343	·0644 ± ·0279
G_l and G_b	− ·0158 ± ·0321	·0263 ± ·0322	·0421 ± ·0213
L and G_l	− ·0289 ± ·0104	·0091 ± ·0100	·0380 ± ·0073
B and G_l	− ·0376 ± ·0325	·0265 ± ·0324	·0641 ± ·0219
I and L	− ·0293 ± ·0204	− ·0534 ± ·0204	− ·0241 ± ·0121
I and B	− ·0496 ± ·0335	− ·1052 ± ·0347	− ·0556 ± ·0245
I and G_l	− ·0664 ± ·3730	− ·0972 ± ·0376	− ·0308 ± ·0233

Omitting the index (I) correlations, we see that only one of the differences between 1914 and 1913 is significant having regard to its probable error, i.e. that of L and G_l; not one of the differences of 1913 and 1920 can be considered significant, but certainly three of the differences of 1914 and 1920 are significant. It is very improbable that the four fundamental correlations of 1914 should all be

below those of 1920 by amounts of twice or more times the probable error. We see that 1920 does not differ significantly from 1913, and we must conclude that 1914 was a year of low organic correlations. It would seem probable that (i) the colony was more stringently selected in 1914 than in 1920, i.e. the birds were more uniform in age or condition, or (ii) that in 1914 there were environmental changes in climate or food-supply during the laying season*.

If we turn to the index correlations with length, breadth and longitudinal girth, we cannot definitely assert a differentiation between 1913 and 1914, but there certainly appears to be such a difference between 1913 and 1920, and possibly between 1914 and 1920.

At first sight it may seem curious that if the correlations of the absolute lengths are not significantly different in 1913 and 1920, the correlations of the index with these absolute lengths can be so. But if we take, for example, the index and breadth, we have

$$r_{B/L,\,B} = \frac{v_B/v_L - r_{BL}}{\sqrt{1 + (v_B/v_L)^2 - 2\,(v_B/v_L)\,r_{BL}}}, \text{ approximately.}$$

Thus the correlation of I and B does not depend only on r_{BL}, but also on the ratio v_B/v_L, where v_B and v_L are the coefficients of variation of breadth and length. Now v_B/v_L was ·6856 in 1920 and ·7673 in 1913. The change in the relative variabilities in the two years was amply sufficient to make a considerable change in $r_{I,B}$ although $r_{B,L}$ remained practically the same. This will account for the apparent paradox.

Since the formula for the product-moment correlation involves the standard deviations in the denominator, it is often assumed that increase of variability reduces correlation. This is, however, a fallacy; increase of variability must of course make itself also felt in the product-moments, and this may well counterbalance the increase in the standard deviations. Changes in *relative* variability will, however, make themselves felt in correlations of indices with absolute lengths, and this point is well illustrated in the present case.

On the whole while there are seasonal changes of some importance in the correlations, there do not appear to be any secular changes in progress, and the organic correlations of the eggs of 1920 are nearer to those of 1913 than to those of 1914.

(8) *The Organic Correlations of Mottling with Ground-Colour.*

This was first worked out as an 11 × 19 contingency table, no categories being grouped. This naturally gave $\bar{\phi}^2$ and $\bar{C}_2$ very high. We have in fact:

$$\phi^2 = 171{,}655, \quad \bar{\phi}^2 = \cdot 220{,}573 \pm \cdot 0146,$$
$$C = \cdot 1465, \qquad \bar{C}_2 = \cdot 4251.$$

From this standpoint ϕ^2 is considerably less than the mean value of ϕ^2 without any association, or there is no significant relationship. Four other tables were

* It must be remembered that 1913 was the hard season, and this might lead to the birds of 1914 being a more stringently selected group.

then formed, namely between amount of Green in Ground-Colour and amount of Brown in Ground-Colour and (i) nature of markings, fine, coarse, confluent, and (ii) position of belt, non-existent, zonal or at top of egg. These tables were the following:

TABLE XVII.

Ground-Colour and Mottling.

(α) Green in Ground-Colour. (β) Brown in Ground-Colour.

Mottling Spots	*1*	*2*	*3*	*4*	Totals	0	*A*	*B*	*C*	*D*	Totals
Fine (*a*+*b*+*c*+*d*)	12·5	101·5	305·5	100·5	520	43·5	79	120	152	125·5	520
Coarse (*e*+*f*+*g*+*h*)	6	84	202·5	64	356·5	32·5	63	75	83	103	356·5
Confluent (*i*+*j*+*k*)	1	13·5	38	14	66·5	5·5	8·5	14·5	20	18	66·5
Totals ...	19·5	199	546	178·5	943	81·5	150·5	209·5	255	246·5	943

(γ) Green in Ground-Colour. (δ) Brown in Ground-Colour.

Mottling Belt	*1*	*2*	*3*	*4*	Totals	0	*A*	*B*	*C*	*D*	Totals
Non-existent (*a*+*e*+*i*)	7	91	233·5	90·5	422	30·5	64·5	96	123	108	422
Zonal (*b*+*c*+*f*+*g*+*j*+*k*)	9·5	91·5	279·5	72·5	453	46·5	76	101·5	108	121	453
At Top (*d*+*h*)	3	16·5	33	15·5	68	4·5	10	12	24	17·5	68
Totals ...	19·5	199	546	178·5	943	81·5	150·5	209·5	255	246·5	943

(α) gives: $\phi^2 = \cdot 002{,}999$, $\bar{\phi}^2 = \cdot 011{,}665 \pm \cdot 0052$,
$C_2 = \cdot 0547$, $\bar{C}_2 = \cdot 1075 \pm \cdot 0121$,
while C_2 corrected for class indices is only ·0712.

(β) gives: $\phi^2 = \cdot 007{,}089$, $\bar{\phi}^2 = \cdot 013{,}786 \pm \cdot 0058$,
$C_2 = \cdot 0839$, $\bar{C}_2 = \cdot 1167 \pm \cdot 0121$,
while C_2 corrected for class indices is only ·1042.

(γ) gives: $\phi^2 = \cdot 009{,}935$, $\bar{\phi}^2 = \cdot 011{,}665 \pm \cdot 0052$,
$C_2 = \cdot 0990$, $\bar{C}_2 = \cdot 1075 \pm \cdot 0121$,
while C_2 corrected for class indices is only ·1265.

(δ) gives: $\phi^2 = \cdot 006{,}877$, $\bar{\phi}^2 = \cdot 013{,}786 \pm \cdot 0058$,
$C_2 = \cdot 0828$, $\bar{C}_2 = \cdot 1167 \pm \cdot 0121$,
while C_2 corrected for class indices is only ·1010.

It is thus clear that none of these methods of approaching the problem shows any significant association between the nature of the mottling and the intensity of the ground-colour, and thus the 1920 results fully confirm those of 1914. The deposition of the mottling is in intensity and position wholly independent of the intensity of both ground-colours; there appears to be no correlation between the process by which the ground-colour arises and the apparatus by means of which the mottling is added.

(9) *Cross Organic Correlations between Pigmentation Characters and Size and Shape Factors.*

We have already (pp. 303—6) considered at some length the relation of Breadth to pigmentation. Breadth will only be included here for the sake of completeness.

(i) Mottling, Length, Longitudinal Girth, Breadth and Index. The following table gives the results to be interpreted for the population of 950 eggs.

TABLE XVIII.

Mottling Category	Number in Array	Means of Arrays			
		Length	Longitudinal Girth	Breadth	Index
a	233·5	4·186	11·367	3·063	73·271
b	122·5	4·144	11·292	3·048	73·644
c	128·5	4·204	11·366	3·037	72·359
d	38	4·191	11·386	3·048	72·661
e	153	4·202	11·432	3·073	73·270
f	118	4·194	11·425	3·066	73·119
g	58	4·173	11·356	3·045	73·036
h	30	4·235	11·483	3·066	72·450
i	38·5	4·143	11·297	3·052	73·911
j	22·5	4·162	11·239	3·016	72·594
k	7·5	4·210	11·485	3·087	73·617
Means of Population		4·185	11·375	3·057	73·125

The correlation ratios of egg size and shape on character of mottling were first determined without any grouping, as this seemed a suitable first test, and eleven arrays did not suggest a need for a class index correction. The results obtained were as follows:

Correlation Ratios of Size and Shape on Mottling.

Character	$\bar{\eta}$	η	Remarks
Length	·1026	·1324 ± ·0215	Hardly significant
Longitudinal Girth .	·1026	·1553 ± ·0214	Possibly just significant
Breadth	·1026	·1629 ± ·0213	Probably significant
Index	·1026	·1384 ± ·0215	Non-significant

Length and Index do not appear to give associations of significance on the present data; Breadth does give a significant association and possibly Longitudinal Girth, which of course involves the breadth.

We can put together the present results with those of earlier censuses and we have the following table:

Correlation Ratios for Mottling, Shape and Size.

	Census 1913	Census 1914		Census 1920	
Mottling and Length	—	·0937	$\bar{\eta}$ = ·0850 ± ·0203	·1324	$\bar{\eta}$ = ·1026 ± ·0217
„ and Longitudinal Girth	—	—		·1553	
„ and Breadth	·1803	·1753		·1629	
„ and Index	·1550	·1598		·1384	

Clearly the census of 1914 confirms that of 1920 for Length, and both those of 1913 and 1914 confirm and emphasise that of 1920 for Breadth, but they indicate a more marked relation than appears in 1920 between Mottling and Index. We have cut up a sheet of the mottling scale and arranged the eggs in the order suggested in the following table, which places them in the magnitude of the characters concerned, but in the case of Length, Longitudinal Girth and Index it did not seem possible by any slight modification of the order to reach an intelligible Mottling classification. In the case of Breadth the matter stands a little

Mottling Classes arranged according to Size of Character.

Length	Longitudinal Girth	Breadth	Index
i	*j*	*j*	*c*
b	*b*	*c*	*h*
j	*i*	*g*	*j*
g	*g*	*d*	*d*
a	*c*	*b*	*g*
d	*a*	*i*	*f*
f	*d*	*a*	*e*
e	*f*	*f*	*a*
c	*e*	*h*	*k*
k	*h*	*e*	*b*
h	*k*	*k*	*i*

differently, *j*, *c*, *g*, *d* and *b* have all belted mottling; *i*, *a*, *e* and *k* have all no belt, while types *f* and *h*—the former coarse mottling with slight zonal belt and the latter coarse mottling with top belt—might easily be confused by a fieldworker with non-belted types. Starting now with two groups as suggested by Breadth, *j* + *g* + *c* + *b* + *d* as belted and *i* + *a* + *e* + *k* + *f*? + *h*? as unbelted, we find that for Index *h* has been introduced into the belted and *b* rejected, for Longitudinal Girth *i* has been introduced and *d* rejected, while for Length *i* and *a* have been introduced and *c* and *d* rejected. These changes are not very great and it is quite

possible that they are largely due to random errors of sampling. It seems reasonable therefore to suggest that the relation of these size characters to Mottling depends on the eggs of smaller size having a belted mottling. Proceeding on these lines we form the following tetrachoric table:

Breadths	Belted Mottling	Unbelted Mottling	Totals
Under 3·50 ...	191·5	234·5	426
3·50 and over ...	178·0	346·0	524
Totals ...	369·5	580·5	950

The value of the tetrachoric correlation is

$$r = \cdot1790 \pm \cdot0342,$$

the greater breadths being associated with the eggs without belts.

It does accordingly seem as if the mechanism by which the mottling was achieved is influenced to a small but significant extent by the dimensions of the egg, an egg of smaller size having a more marked belt. This is in general accordance with the results obtained in earlier years, and with the views already expressed on pp. 305—6.

(ii) Ground-Colour, Length, Breadth and Index. The following table gives the mean values for arrays of equal density of pigmentation in 947 eggs.

TABLE XIX.

Ground-Colour and Size and Shape.

Ground-Colour Value	Number in Array	Means of Arrays		
		Length	Breadth	Index
$w_2 = 2 + A_1$	11·5	4·257	3·068	72·602
$w_3 = 3 + A_2 + B_1$ ...	76·5	4·218	3·074	73·042
$w_4 = 4 + A_3 + B_2 + C_1$.	174·5	4·177	3·049	73·162
$w_5 = A_4 + B_3 + C_2 + D_1$	240·5	4·194	3·054	72·796
$w_6 = B_4 + C_3 + D_2$ ...	201·5	4·174	3·053	73·347
$w_7 = C_4 + D_3$	186	4·182	3·058	73·138
$w_8 = D_4$	56·5	4·172	3·065	73·552
Means of General Population		4·186	3·056	73·111

On p. 305 we have already seen that there is no significance in the correlation ratio for Breadth and value of Ground-Colour. For:

Length, we have: $\eta = \cdot0879$, while $\bar{\eta} = \cdot0796 \pm \cdot0218$.

Index, „ $\eta = \cdot0820$ „ $\bar{\eta} = \cdot0796 \pm \cdot0218$.

Thus we see that none of the correlations is significant.

In order to meet the suggestion that total value might not be the factor of Ground-Colour which was of importance, the values of the correlation ratio were determined using all the 19 categories of Ground-Colour, and we found:

For Length, $\eta = \cdot1511$, while $\bar{\eta} = \cdot1379 \pm \cdot0215$.

For Breadth, $\eta = \cdot1355$ „ $\bar{\eta} = \cdot1379 \pm \cdot0215$.

For Index, $\eta = \cdot1287$ „ $\bar{\eta} = \cdot1379 \pm \cdot0215$.

Thus the results were again without significance, or, we conclude—in accordance with the result of the 1914 census, even if the approach has been by a different path—that there is no association of Ground-Colour with either the shape or the size of the egg.

(iii) We may finish this section on the organic relations by considering the changes in the regression coefficients. These are given with their probable errors in Table XX below.

TABLE XX.

Secular Changes in Regression Coefficients.

Characters	1913	1914	1920
G_l on L	1·8390 ± ·0390	1·6109 ± ·0201	1·7559 ± ·0197
G_l on B	1·9810 ± ·1275	1·7111 ± ·0626	2·1402 ± ·0715
B on L	·1221 ± ·0211	·1126 ± ·0106	·1382 ± ·0105
B on I	·0144 ± ·0010	·0116 ± ·0005	·0111 ± ·0006
G_l on I	− ·0418 ± ·0040	− ·0452 ± ·0018	− ·0523 ± ·0021
L on G_l	·4215 ± ·0089	·4501 ± ·0056	·4505 ± ·0051
B on G_l	·1373 ± ·0088	·1369 ± ·0060	·1404 ± ·0047
L on B	·4036 ± ·0697	·3932 ± ·0370	·5436 ± ·0417
I on B	17·5342 ± 1·1842	15·9437 ± ·7612	14·2838 ± ·7224
I on G_l	− 3·5150 ± ·3333	− 4·4690 ± ·2158	− 4·4143 ± ·1759

In considering this table we have to remember that if there be two variates, x and y, and if x only has been selected, then the regression coefficient of y on x will not be changed. We therefore conclude that:

(i) For pair G_l and L. Both must have been selected as between 1913 and 1914, but only G_l as between 1914 and 1920.

(ii) For pair G_l and B. There is no evidence of selection of B in the B on G_l returns for the three seasons, but considerable evidence of the selection of G_l.

(iii) For pair B and L. There is no evidence of selection of either B or L as between 1913 and 1914; but some evidence of selection of B and more of selection of L as between 1914 and 1920.

(iv) For pair I and B. There is some evidence of selection of both I and B as between 1913 and 1914; there is no evidence of selection of B, and but slight of the selection of I, as between 1914 and 1920.

(v) For pair I and G_l. There is no evidence of the selection of either I or G_l as between 1913 and 1914. There is evidence of the selection of G_l but not of I as between 1914 and 1920.

To sum up, for the three seasons we see that there is considerable evidence of the Longitudinal Girth being a character subject to considerable seasonal selection; there is also some evidence of the Index being selected, but not so marked of the absolute Lengths or Breadths. Longitudinal Girth is a reasonable measure of the bulk of the egg, and Index of the shape. We should therefore be inclined to say that seasonal variation in the correlation of the egg characters appears to be chiefly influenced by seasonal changes in the variability of bulk and shape. This seems to be reasonably in accord with the values for the standard deviations in Table IV. For we see how significant are the changes in Longitudinal Girth, Index and Bulk between 1914 and 1920. At the same time there have been significant changes in Length and Breadth, and these are evidenced in (iii) above.

In the earlier papers we gave the full regression formula for G_l on B and L, and we now complete the series with that for 1920. Thus we have:

$$1913, \quad G_l = 1{\cdot}2701B + 1{\cdot}6415L + {\cdot}8224,$$
$$1914, \quad G_l = 1{\cdot}1273B + 1{\cdot}4840L + 1{\cdot}9180,$$
$$1920, \quad G_l = 1{\cdot}2826B + 1{\cdot}5773L + {\cdot}8522.$$

The formula for 1920 is reasonably in accord with that of 1913, but both differ considerably from that of 1914, which confirms the view that there was considerable selection by girth in 1914 compared with the other two seasons. However when the 1913 and 1914 are applied to predict the girth of 1920 eggs, they accord more closely with themselves and both give values in excess of the observed. This is brought out in the following table for ten typical eggs.

TABLE XXI.

Observed and Theoretical Girths.

Observed	1920 Formula	1913 Formula	1914 Formula
10·67	10·76	10·94	11·01
11·05	11·08	11·27	11·30
11·12	11·19	11·38	11·40
11·25	11·30	11·50	11·51
11·30	11·27	11·47	11·48
11·35	11·30	11·49	11·50
11·50	11·51	11·72	11·71
11·60	11·61	11·82	11·80
11·80	11·81	12·03	11·98
11·88	11·73	11·94	11·90

(10) *The Homotypic Correlations.*

(i) Direct Homotyposis in Size and Shape. The following table gives the correlations in size and shape of eggs from the same clutch. The important point

is now obvious: that 1913 and not 1914 was the anomalous year. Speaking generally, while (lengths excepted) the correlations of 1920 have fallen below those

TABLE XXII.

Direct Homotyposis in Size and Shape.

Character	Correlations		
	1913	1914	1920
Lengths	·4643 ± ·0346	·6056 ± ·0107	·6591 ± ·0097
Breadths	·5176 ± ·0326	·7327 ± ·0078	·6170 ± ·0107
Longitudinal Girths ...	·5076 ± ·0327	·6689 ± ·0093	·6271 ± ·0105
Transverse Girths ...	·4621 ± ·0350	·7469 ± ·0075	·6416 ± ·0102
Indices	·5537 ± ·0308	·5327 ± ·0120	·4957 ± ·0130

of 1914, they are all (Index excepted) very substantially above those of 1913. With the exceptions of Lengths and Indices we see that the correlations are not changing in the same direction, and we feel bound to conclude that there exists some environmental factor which must in a considerable measure determine from year to year the degree of likeness between eggs of the same clutch. At first it seemed possible that the different proportions of two and three egg clutches in the different years might contribute to the result. But on more mature consideration this should be a factor affecting all characters alike, while it is clear that the seasonal influence, whatever it may be, has a relationship to the characters which differs from one to another. Likeness in the eggs of the same clutch should indicate likeness in the condition of the egg-producing organ during the period of laying. It is conceivable that the intervals between laying of the individual eggs may be longer in one year than another, so that there is more time for constitutional changes to take place. Of course a large proportion of young birds, first layers, would probably weaken the homotyposis*, but the values for 1914 and 1920 already appear on theoretical grounds too high. Much light might be thrown on the matter, were a daily record of barometer, thermometer and anaemometer taken during the laying period.

(ii) Direct Homotyposis in Pigmentation. We may as before consider the Mottling from two standpoints, density of belt or fineness of markings, and Ground-Colour may be dealt with in three ways, the intensity of green, the intensity of brown or simply the value without regard to colour. Tables XXIII and XXIV give the data for value in Ground-Colour and Density of Belt. As far as Mottling is concerned the homotypic correlations are not quite homologous, those for 1913 and 1914 were obtained by an attempt to appreciate the proportion of area mottled, while that for 1920 was by the scale passing from fine to confluent markings. There is little doubt, however, of a high correlation between the two scales. All the results were obtained by mean square contingency. We trace

* The first egg is often a "reckling"—a feeble, dwarf egg. "There is often a reckling in a cletch"—Rutlandshire.

TABLE XXIII.

Direct Homotyposis in Value of Ground-Colour.

Value of Ground-Colour of First Egg.

Value of Ground-Colour of Second Egg.	w_2	w_3	w_4	w_5	w_6	w_7	w_8	Totals
w_2	3	8	2	1·75	1·5	·75	—	17
w_3	8	44	40	19	6	6	—	123
w_4	2	40	135	68	29·5	12	2	288·5
w_5	1·75	19	68	153	95·25	41	9	387
w_6	1·5	6	29·5	95·25	101·5	75·75	13	322·5
w_7	·75	6	12	41	75·75	126·5	28	290
w_8	—	—	2	9	13	28	30	82
Totals	17	123	288·5	387	322·5	290	82	1510

Class Index Correlation = ·9743.

TABLE XXIV.

Direct Homotyposis in Density of Belt.

Density of Belt of First Egg.

Density of Belt of Second Egg.	$a+e+i$	b	$c+d$	f	$g+h$	$j+k$	Totals
$a+e+i$	389	79·5	85·75	63	54·5	16·25	688
b	79·5	34	55·75	11·5	10·75	2·5	194
$c+d$	85·75	55·75	62·5	22	13	5	244
f	63	11·5	22	52	30	13·5	192
$g+h$	54·5	10·75	13	30	30	7·25	145·5
$j+k$	16·25	2·5	5	13·5	7·25	4	48·5
Totals	688	194	244	192	145·5	48·5	1512

Class Index Correlation = ·9132.

TABLE XXV.

Seasonal Differences of Homotyposis in Pigmentation.

Character of Pigmentation	1913	1914	1920
Mottling: Density of Belt	—	—	$C_2 = ·4417$ *
" Fineness of Markings ...	$C_2 = ·3500$	$C_2 = ·6267$	$C_2 = ·6406$
Ground-Colour: Intensity of Green	—	—	$C_2 = ·7791$
" " " Brown	—	—	$C_2 = ·6782$
Values	$C_2 = ·5709$	$C_2 = ·7480$	$C_2 = ·6520$

* To show how significant these values really are, we give the $\overline{C}_2$ values with their probable errors in order:

$\overline{C}_2 = ·1371 \pm ·0133$, $\overline{C}_2 = ·0726 \pm ·0123$, $\overline{C}_2 = ·0990 \pm ·0124$, $\overline{C}_2 = ·1251 \pm ·0121$, $\overline{C}_2 = ·1755 \pm ·0118$.

exactly the same result as in the size and shape homotypic correlations—they vary from season to season and those of 1913 are very much lower than those of 1914 or 1920. The relatively low value for the homotyposis in the case of Density of Belt suggests that the temporary condition of the oviduct largely affects this character.

Dr Katherine Watson, who was a fieldworker not only in 1920 but in 1913, suggests that (i) the workers may not have been so experienced in the first year of their work as in later years, and (ii) the data were in 1913 largely collected towards the end of the second egg-laying period, and that when the second batch began to hatch out a good number of eggs were addled and washed out of the nests, and may have been wrongly recorded as belonging to the same clutch. We cannot readily accept these explanations of the low homotyposis of 1913, for if it was reduced by wrong recording, then the reduction must have occurred in *all* characters, but the Index homotyposis is actually higher in 1913 than in 1914 or 1920. The true explanation must be one which accounts for a differential effect*. We should doubtless have clearer ideas, if there had not been a total omission of observation during the war years. At the same time the recording of the characters, their tabling and the resulting computations are of such a laborious character that it would be exceedingly difficult to carry out an adequate annual survey and its reduction for a long series of years.

(iii) Cross Homotyposis in Size Characters. The following table gives the values of the cross homotypic correlations in size characters.

TABLE XXVI.

Cross Homotypic Size Correlations.

Characters of the two Eggs from Clutch	1913	1914	1920
Length and Breadth	·922 ± ·0441	·2621 ± ·0157	·4191 ± ·0142
Longitudinal and Transverse Girths ...	·2603 ± ·0413	·4546 ± ·0134	·4336 ± ·0140
Length and Longitudinal Girth ...	·4229 ± ·0362	·5854 ± ·0111	·6372 ± ·0102
Breadth and Longitudinal Girth ...	·2530 ± ·0416	·4162 ± ·0140	·4242 ± ·0141

Here again we are faced in Cross Homotyposis with the same result as we found for Direct Homotyposis, namely the low values of 1913 as compared with those for either 1914 or 1920. It will, perhaps, be remembered that one of us has suggested† that

Cross Homotypic correlation of A and B should equal $\frac{1}{2}$ (direct correlation of A with A + direct correlation of B with B) × (organic correlation of A with B)

This was found to give fairly satisfactory results for 1913, but for 1920 like 1914 there is no reasonable accordance.

* Actually data in our possession for sparrows', thrushes' and other birds' eggs show a very similar result, i.e. a very considerable constancy of the index correlations, while the absolute size correlations vary not only with the season, but from locality to locality in the same season.

† Pearson, *Phil. Trans.* Vol. 197 A, p. 290.

We have indeed the following table:

TABLE XXVII. *Calculated and Observed Cross Homotyposis.*

Characters of two Eggs of same Clutch	1913		1914		1920	
	Calculated	Observed	Calculated	Observed	Calculated	Observed
Length and Breadth	·1090	·0922	·1408	·2621	·1753	·4191
Longitudinal and Transverse Girths	·2568	·2603	·3638	·4546	·3801	·4336
Length and Longitudinal Girth	·4278	·4229	·5426	·5854	·5726	·6372
Breadth and Longitudinal Girth	·2674	·2530	·3392	·4162	·3407	·4242

In 1914 and 1920 the observed is in every case in excess of the calculated. One would like to consider that 1913 represented a normal year, but there is no justification for assuming that 1914 and 1920 are abnormal. It would be more reasonable to suppose that birds' eggs cannot be taken as pure homotypes, i.e. as the random result of mere cell-division. They are this together with additional characterisation impressed on the egg after it leaves the ovary and before it is laid, this latter phase of the production being subject to seasonal influence of some one kind or another. This is scarcely an explanation, but the variation with the season in homotyposis needs some further inquiry: what are the conditions influencing the egg-layer which make the eggs of the clutches more alike in one year than in another year? We should be much surprised if the resemblance between puppies of the same litter were more intense in one year than in a second. There are additional factors coming in with the coating of the birds' eggs, how are they affected by environmental differences season by season?

(iv) Cross Homotyposis in Pigment Characters. We have dealt here only with mottling as measured by fineness or coarseness of blotches and the intensity of ground-colour. We find:

Mottling and Green in Ground-Colour:

$$C_2 = \cdot 0779,$$
$$\overline{C}_2 = \cdot 0857 \pm \cdot 0128.$$

Mottling and Brown in Ground-Colour:

$$C_2 = \cdot 1234,$$
$$\overline{C}_2 = \cdot 0955 \pm \cdot 0134.$$

There is thus certainly no relation of a significant kind between Mottling and Green Ground-Colour. There might possibly be some small association of coarser mottling of one egg with intenser brown of another, but the difference of C_2 and $\overline{C}_2$ is only about twice the probable error, and as the organic relation between Mottling and Brown Ground-Colour in the *same* egg is not significant, it seems reasonable to suppose this cross-homotyposis is zero. This is in accordance with the 1913 and 1914 results where with a different measure of Mottling and using 'value' of Ground-Colour we found:

1913	1914
$C_2 = \cdot3989,$	$C_2 = \cdot1479,$
$\bar{C}_2 = \cdot3169 \pm \cdot0451,$	$\bar{C}_2 = \cdot1992 \pm \cdot0169,$

$\bar{C}_2$ being actually greater than C_2 in 1914. We have worked out the cross-homotyposis of intensities of brown and of green ground-colours in pairs of eggs from the same clutch. It is given by

$$C_2 = \cdot3603, \quad \text{while} \quad \bar{C}_2 = \cdot1115 \pm \cdot0123;$$

and hence the cross-homotyposis is significant. The organic correlation between intensities of green and brown in the same egg has been worked out and it was found to be

$$C_2 = \cdot2915, \quad \bar{C}_2 = \cdot1438 \pm \cdot0150.$$

Accordingly C_2 is significant. Pearson's relation would give

$$\tfrac{1}{2}(\cdot7791 + \cdot6782) \times \cdot2915 = \cdot2124,$$

instead of the observed ·3603.

It is accordingly clear, as in the case of cross-homotyposis in size characters, that the formula provides values which fall below the observed—at any rate in certain seasons. It must be admitted that the 1920 data, while confirming those of 1914 in several points where they differed from 1913, have not made any clearer the homotypic difficulties. It is a very startling idea to think that the individuality which the egg layer impresses upon her eggs is a subject for seasonal variation, that she should in one season lay eggs more alike than in another season. As was said in the second cooperative memoir on the tern's egg, this fact should give rise to much hard thinking. Is it possible that environmental factors can modify not only the degree of resemblance between two eggs of the same bird, but the degree of resemblance between two individuals proceeding from those eggs?—The one, of course, does not follow from the other, but the former renders the latter more conceivable. It would be an interesting inquiry to determine in the case of large families whether the eldest child more closely resembles the second, than a child born a long time after. Are two puppies from the same litter more alike than two puppies from different litters with the same dam and sire? If they should turn out to be so, then we must suppose, either that the germ plasm is modified by age or environmental conditions, or that the selections of germ plasm available on two different occasions may not be random samples from the whole stock.

We feel certain that there is much still to be learnt from a secular examination of the Blakeney Ternery—birds, eggs, nests and environments—but the work already done shows that it must be persistent, namely annual and at the first nesting time. The problem then turns on whether volunteers could be found for the work—say for a period of five to ten years—not only for the fieldwork but for the laborious annual reductions. It is well worth doing for we see from the results already obtained that we are far from dealing with a stable population, and only such persistent investigation could determine whether the marked variations have a definite trend, or are merely seasonal. From the standpoint of evolution such a determination would we hold be of great interest.

APPENDIX. TABULATED DATA.

(i) *Type and Environment of Nest with Egg Characters.*

Table A. Type of Nest and Length of Egg.
" B. Environment of Nest and Length of Egg.
" C. Type of Nest and Breadth of Egg.
" D. Environment of Nest and Breadth of Egg.
" E. Type of Nest and Index of Ovality.
" F. Environment of Nest and Index of Ovality.
" G. Type of Nest and Bulk of Egg.
" H. Environment of Nest and Bulk of Egg.
" I. Type of Nest and Longitudinal Girth of Egg.
" J. Environment of Nest and Longitudinal Girth of Egg.
" K. Type of Nest and Index ($100B/L$) of Egg.
" L. Environment of Nest and Index ($100B/L$) of Egg.
" M. Type of Nest and Mottling of Egg.
" N. Environment of Nest and Mottling of Egg.
" O. Type of Nest and Categories of Ground-Colour of Egg.
" P. Environment of Nest and Categories of Ground-Colour of Egg.
" Q. Type of Nest and Type of Environment.
" R. Type of Nest and Type of Environment weighted with Number of Eggs in Clutch.

(ii) *Organic Correlations.*

" S. Organic Correlation of Length and Breadth of Egg.
" T. " " Breadth and Longitudinal Girth of Egg.
" U. " " Length and Index of Egg.
" V. " " Length and Longitudinal Girth of Egg.
" W. " " Longitudinal and Transverse Girths of Egg.
" X. " " Longitudinal Girth and Index ($100B/L$) of Egg.
" Y. " " Breadth and Index ($100B/L$) of Egg.
" Z. " " Mottling and Breadth of Egg.
" AA. " " Ground-Colour and Breadth of Egg.
" BB. " " Mottling and Length of Egg.
" CC. " " Mottling and Longitudinal Girth of Egg.
" DD. " " Mottling and Index ($100B/L$) of Egg.
" EE. " " Ground-Colour and Length of Egg.
" FF. " " Ground-Colour and Index ($100B/L$) of Egg.
" GG. " " Ground-Colour and Mottling of Egg.

(iii) *Direct Homotyposis of Eggs of same Clutch.*

" HH. Direct Homotyposis of Longitudinal Girths.
" II. " " Transverse Girths.
" JJ. " " Lengths.
" KK. " " Indices ($100B/L$).
" LL. " " Breadths.
" MM. " " Mottlings.
" NN. " " Ground-Colours.

(iv) *Cross-Homotyposis of Eggs of same Clutch.*

" OO. Cross-Homotyposis of Longitudinal and Transverse Girths.
" PP. " " Length and Longitudinal Girth.
" QQ. " " Breadth and Longitudinal Girth.
" RR. " " Length and Breadth.
" SS. " " Mottling and Ground-Colour.

APPENDIX. TABULATED DATA.

(i) Type and Environment of Nest with Egg Characters.

TABLE A. *Type of Nest and Length of Egg.*

Length.

Type of Nest.	3·55—	3·60—	3·65—	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
n_0	—	—	—	—	—	1	1	1	2	8	6	7	10	5	6	8	7	4	—	1	—	—	—	67
n_1	—	—	—	1	1	1	3	4	11	10	20	31	23	29	22	28	14	5	6	5	2	—	2	218
n_2	—	—	—	1	2	4	4	12	18	25	23	42	35	29	26	22	23	16	14	5	4	1	2	308
n_3	—	—	—	—	3	6	5	9	16	14	21	29	15	18	21	26	8	6	3	3	2	1	1	207
n_4	1	—	—	—	—	2	3	4	8	12	14	17	17	17	16	22	3	9	2	1	4	—	—	152
*	—	—	—	—	—	—	—	—	—	—	1	2	1	—	—	—	—	—	—	—	—	—	—	4
Totals	1	0	0	2	6	14	16	30	55	69	85	128	101	98	91	106	55	40	25	15	12	2	5	956

TABLE B. *Environment of Nest and Length of Egg.*

Length.

Environment of Nest.	3·55—	3·60—	3·65—	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
e_1	—	—	—	—	1	—	1	4	5	11	15	16	17	15	7	15	10	3	4	1	3	1	—	129
e_2	1	—	—	—	—	—	—	2	7	9	12	24	22	19	13	33	11	9	7	2	1	—	1	173
e_3	—	—	—	1	2	4	2	8	6	14	10	15	12	15	12	12	6	6	3	3	4	1	2	138
e_4	—	—	—	1	3	9	10	10	25	21	31	48	28	29	42	28	17	13	9	6	2	—	2	334
e_5	—	—	—	—	—	—	3	3	9	13	10	17	16	15	14	16	8	7	2	2	2	—	—	137
e_6	—	—	—	—	—	1	—	2	3	1	7	8	4	4	3	2	3	2	—	—	—	—	—	40
*	—	—	—	—	—	—	—	1	—	—	—	—	2	1	—	—	—	—	—	1	—	—	—	5
Totals	1	0	0	2	6	14	16	30	55	69	85	128	101	98	91	106	55	40	25	15	12	2	5	956

TABLE C. *Type of Nest and Breadth of Egg.*

Breadth.

Type of Nest.	2·65—	2·70—	2·75—	2·80—	2·85—	2·90—	2·95—	3·00—	3·05—	3·10—	3·15—	3·20—	3·25—	Totals
n_0	—	—	—	—	2	3	8	14	16	9	9	3	3	67
n_1	—	—	—	1	7	14	27	59	42	45	14	8	1	218
n_2	1	3	1	4	5	24	45	60	60	73	19	11	2	308
n_3	—	—	—	3	1	12	22	62	44	44	14	5	—	207
n_4	—	—	1	—	1	5	11	33	29	42	19	7	4	152
*	—	—	—	—	—	—	1	—	2	—	1	—	—	4
Totals	1	3	2	8	16	58	114	228	193	213	76	34	10	956

TABLE D. *Environment of Nest and Breadth of Egg.*

Breadth.

Environment of Nest.	2·65—	2·70—	2·75—	2·80—	2·85—	2·90—	2·95—	3·00—	3·05—	3·10—	3·15—	3·20—	3·25—	Totals
e_1	—	—	—	1	2	12	15	33	28	31	4	2	1	129
e_2	—	—	1	1	—	5	19	35	34	40	24	11	3	173
e_3	1	3	1	3	4	7	11	40	27	33	5	3	—	138
e_4	—	—	—	3	8	21	46	80	59	72	28	14	3	334
e_5	—	—	—	—	2	10	17	25	35	29	14	3	2	137
e_6	—	—	—	—	—	3	5	15	8	6	1	1	1	40
*	—	—	—	—	—	—	1	—	2	2	—	—	—	5
Totals	1	3	2	8	16	58	114	228	193	213	76	34	10	956

* Character unrecorded.

TABLE E. *Type of Nest and Index of Ovality.*

Type of Nest.	Index of Ovality. 46—	48—	50—	52—	54—	56—	58—	60—	62—	64—	66—	68—	Totals
n_0	—	—	3	6	10	13	19	6	7	2	1	—	67
n_1	3	1	13	20	39	56	36	28	16	5	1	—	218
n_2	1	9	14	36	51	70	46	52	15	10	2	2	308
n_3	1	3	6	17	32	42	38	36	19	6	4	3	207
n_4	—	3	3	7	20	39	30	27	11	9	—	3	152
*	—	—	—	—	—	1	1	1	—	1	—	—	4
Totals	5	16	39	86	152	221	170	150	68	33	8	8	956

TABLE F. *Environment of Nest and Index of Ovality.*

Environment of Nest.	Index of Ovality. 46—	48—	50—	52—	54—	56—	58—	60—	62—	64—	66—	68—	Totals
e_1	1	5	2	11	24	32	26	20	6	2	—	—	129
e_2	—	3	6	15	33	42	32	25	8	7	1	1	173
e_3	2	5	8	16	21	24	22	20	11	6	—	3	138
e_4	2	2	13	33	46	68	65	58	30	10	4	3	334
e_5	—	1	10	6	24	41	16	20	10	6	2	1	137
e_6	—	—	—	4	4	13	7	7	3	2	—	—	40
*	—	—	—	1	—	1	2	—	—	—	1	—	5
Totals	5	16	39	86	152	221	170	150	68	33	8	8	956

TABLE G. *Type of Nest and Bulk of Egg.*

Type of Nest.	Bulk (i.e. LB^2). 26—	27—	28—	29—	30—	31—	32—	33—	34—	35—	36—	37—	38—	39—	40—	41—	42—	43—	44—	45—	46—	Totals
n_0	—	—	—	—	—	—	2	2	1	4	5	6	5	13	10	6	5	4	2	1	1	67
n_1	—	—	—	1	1	1	—	2	7	12	27	27	33	27	26	20	13	9	8	4	—	218
n_2	1	1	1	1	—	3	4	7	8	25	23	36	41	39	37	30	21	13	8	8	1	308
n_3	—	—	—	—	1	1	2	2	11	19	16	31	27	34	25	14	10	4	8	2	—	207
n_4	—	1	—	—	—	1	—	3	4	6	7	21	17	13	27	20	13	10	3	6	—	152
*	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	—	—	—	—	—	—	4
Totals	1	2	1	2	2	6	8	16	31	66	78	122	124	127	126	90	62	40	29	21	2	956

TABLE H. *Environment of Nest and Bulk of Egg.*

Environment of Nest.	Bulk (i.e. LB^2). 26—	27—	28—	29—	30—	31—	32—	33—	34—	35—	36—	37—	38—	39—	40—	41—	42—	43—	44—	45—	46—	Totals
e_1	—	—	—	—	2	—	—	1	4	14	10	19	13	19	19	11	11	3	2	1	—	129
e_2	—	1	—	—	—	1	—	1	1	7	10	13	23	27	28	24	16	7	5	9	—	173
e_3	1	1	1	1	—	2	3	2	5	10	8	20	17	17	20	16	7	2	4	1	—	138
e_4	—	—	—	1	—	3	5	8	18	20	33	39	46	41	40	25	16	17	13	8	1	334
e_5	—	—	—	—	—	—	—	3	2	9	14	23	17	16	15	14	9	10	3	1	1	137
e_6	—	—	—	—	—	—	—	1	—	6	3	8	8	5	3	—	3	1	1	1	—	40
*	—	—	—	—	—	—	—	—	1	—	—	—	—	2	1	—	—	—	1	—	—	5
Totals	1	2	1	2	2	6	8	16	31	66	78	122	124	127	126	90	62	40	29	21	2	956

* Character unrecorded.

TABLE I. *Type of Nest and Longitudinal Girth of Egg.*

Longitudinal Girth.

Type of Nest.	9·80—	9·90—	—	10·20—	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
n_0	—	—	·	—	—	—	1	2	—	2	2	6	7	6	6	7	7	7	8	4	1	1	—	—	—	—	67
n_1	—	—	·	—	1	—	2	1	1	7	7	13	18	35	26	34	14	22	12	9	5	5	4	1	—	1	218
n_2	—	—	·	—	1	—	3	4	3	12	11	30	30	35	32	30	36	24	20	19	9	6	1	2	—	—	308
n_3	—	—	·	—	—	2	—	6	5	9	16	16	23	21	29	19	19	12	15	4	6	3	1	—	1	—	207
n_4	1	—	·	—	—	—	—	2	1	5	2	14	11	20	20	11	22	11	11	12	8	1	—	—	—	—	152
*	—	—	·	—	—	—	—	—	—	—	—	—	1	2	1	—	—	—	—	—	—	—	—	—	—	—	4
Totals	1	0	0	0	2	2	6	15	10	35	38	79	90	119	114	101	98	76	66	48	29	16	6	3	1	1	956

TABLE J. *Environment of Nest and Longitudinal Girth of Egg.*

Longitudinal Girth.

Environment of Nest.	9·80—	9·90—	—	10·20—	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
e_1	—	—	·	—	—	1	—	1	1	4	6	16	13	9	16	13	14	14	10	7	2	—	1	1	—	—	129
e_2	1	—	·	—	—	—	—	1	1	—	3	11	8	21	19	19	25	23	13	12	9	5	—	—	1	1	173
e_3	—	—	·	—	1	—	1	2	1	10	6	10	14	17	17	14	12	11	8	4	5	3	2	—	—	—	138
e_4	—	—	·	—	1	1	5	10	6	14	15	29	30	47	35	32	33	20	25	15	7	5	2	2	—	—	334
e_5	—	—	·	—	—	—	—	—	1	5	5	8	20	15	21	19	13	7	7	8	5	2	1	—	—	—	137
e_6	—	—	·	—	—	—	—	1	—	1	3	5	5	10	4	3	1	1	3	2	1	—	—	—	—	—	40
*	—	—	·	—	—	—	—	—	—	1	—	—	—	—	2	1	—	—	—	—	—	1	—	—	—	—	5
Totals	1	0	0	0	2	2	6	15	10	35	38	79	90	119	114	101	98	76	66	48	29	16	6	3	1	1	956

TABLE K. *Type of Nest and Index* (100*B*/*L*) *of Egg.*

Index.

Type of Nest.	62—	64—	66—	68—	70—	72—	74—	76—	78—	80—	82—	Totals
n_0	—	—	1	7	11	20	17	8	3	—	—	67
n_1	1	2	11	21	50	60	41	27	5	—	—	218
n_2	1	5	16	35	61	76	66	33	11	4	—	308
n_3	1	2	5	14	42	49	51	30	8	4	1	207
n_4	—	1	5	7	24	48	38	17	9	2	1	152
*	—	—	—	—	1	—	2	—	1	—	—	4
Totals	3	10	38	84	189	253	215	115	37	10	2	956

TABLE L. *Environment of Nest and Index* (100*B*/*L*) *of Egg.*

Index.

Environment of Nest.	62—	64—	66—	68—	70—	72—	74—	76—	78—	80—	82—	Totals
e_1	1	3	4	10	30	36	32	11	2	—	—	129
e_2	—	1	6	15	42	45	39	16	7	2	—	173
e_3	1	3	11	14	27	28	27	18	6	3	—	138
e_4	1	3	9	34	54	83	82	50	13	4	1	334
e_5	—	—	8	6	32	43	24	15	7	1	1	137
e_6	—	—	—	4	4	17	9	4	2	—	—	40
*	—	—	—	1	—	1	2	1	—	—	—	5
Totals	3	10	38	84	189	253	215	115	37	10	2	956

* Character unrecorded.

TABLE M. *Type of Nest and Mottling of Egg.*

Mottling.

Type of Nest.	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*	Totals
n_0	9·5	10·5	10	5	9·5	11	3	3	2	3·5	—	—	67
n_1	53·5	28	30	10	25·5	25·5	14·5	9	14	8	—	—	218
n_2	66·5	41	35·5	12	50	46·5	25	10	12·5	3·5	3·5	2	308
n_3	65	20	32·5	8	29	23	8·5	4	4·5	7·5	3	2	207
n_4	37	22	19·5	3	39	12	7	4	5·5	—	1	2	152
*	2	1	1	—	—	—	—	—	—	—	—	—	4
Totals	233·5	122·5	128·5	38	153	118	58	30	38·5	22·5	7·5	6	956

TABLE N. *Environment of Nest and Mottling of Egg.*

Mottling.

Environment of Nest.	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*	Totals
e_1	29·5	13	19·5	7	14	22·5	9	3·5	8	1	1	1	129
e_2	31·5	18	22	4	35·5	25·5	12·5	9	7·5	5·5	2	—	173
e_3	32	17	14	7	17	15·5	14	6·5	5·5	6	1·5	2	138
e_4	80	53	55·5	13	59·5	30	12·5	8	13	5·5	3	1	334
e_5	44·5	17·5	12·5	4	21·5	19·5	7	2	4·5	2	—	2	137
e_6	13·5	4	5	3	5	4	3	1	—	1·5	—	—	40
*	2·5	—	—	—	·5	1	—	—	—	1	—	—	5
Totals	233·5	122·5	128·5	38	153	118	58	30	38·5	22·5	7·5	6	956

TABLE O. *Type of Nest and Categories of Ground-Colour of Egg.*

Categories of Ground-Colour.

Type of Nest.	*2*	*3*	*4*	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	D_1	D_2	D_3	D_4	*	Totals
n_0	1	1	4	1	2	6	1	1	2	16	2	—	8	6	3	1	—	10	2	—	67
n_1	1	12	8	—	5	22·5	3	—	10	40	7·5	2	15	30·5	5	—	4·5	40	11	1	218
n_2	5·5	13	11	—	11	29·5	8·5	5·5	10	36·5	11	4	25·5	49·5	16·5	1	8·5	42·5	17	2	308
n_3	1	3	7	—	10	15	9	1	8·5	23	2	1	18·5	36	7·5	1	14	31	14·5	4	207
n_4	2	7	5	—	4	21	2	—	7	19·5	5	—	11	12	5	—	12	25·5	12	2	152
*	—	—	—	—	—	—	—	1	1	—	—	—	1	1	—	—	—	—	—	—	4
Totals	10·5	36	35	1	32	94	23·5	8·5	38·5	135	27·5	7	79	135	37	3	39	149	56·5	9	956

TABLE P. *Environment of Nest and Categories of Ground-Colour of Egg.*

Categories of Ground-Colour.

Environment of Nest.	*2*	*3*	*4*	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	D_1	D_2	D_3	D_4	*	Totals
e_1	2	9	2	—	5	11	6	1	9	14	1	—	12·5	20·5	7·5	—	3·5	18	5	2	129
e_2	2	8	5	—	8	26	4	4	11	24	6	5	10	16	8	1	6	17	9	3	173
e_3	2	2	3	—	3	15·5	·5	1	6	21·5	1	—	13	20	5	1	7	27·5	5	4	138
e_4	3	10	17	1	14	21·5	7	·5	3	53	11·5	—	24	58	9	1	16·5	56	28	—	334
e_5	1·5	3	5	—	2	15	5	—	7·5	14·5	7	1	16·5	13·5	4·5	—	5	26·5	9·5	—	137
e_6	—	3	3	—	—	5	1	2	2	7	1	1	3	6	1	—	1	4	—	—	40
*	—	1	—	—	—	—	—	—	—	1	—	—	—	1	2	—	—	—	—	—	5
Totals	10·5	36	35	1	32	94	23·5	8·5	38·5	135	27·5	7	79	135	37	3	39	149	56·5	9	956

* Character unrecorded.

TABLE Q. *Type of Nest and Type of Environment.*

Type of Nest.

Environment of Nest.	n_0	n_1	n_2	n_3	n_4	*	Totals
e_1	1	12	14	20	9	—	56
e_2	5	15	31	7	14	—	72
e_3	8	9	24	9	6	1	57
e_4	9	43	45	33	16	—	146
e_5	8	16	10	11	15	1	61
e_6	3	3	3	6	—	—	15
*	—	3	—	—	—	—	3
Totals	34	101	127	86	60	2	410

TABLE R. *Type of Nest and Type of Environment weighted with Number of Eggs in Clutch.*

Type of Nest.

Env. of Nest of Egg.	n_0	n_1	n_2	n_3	n_4	*	Totals
e_1	1	25	32	50	21	—	129
e_2	11	32	74	20	36	—	173
e_3	16	22	60	23	14	3	138
e_4	12	95	110	73	44	—	334
e_5	19	32	23	25	37	1	137
e_6	8	7	9	16	—	—	40
*	—	5	—	—	—	—	5
Totals	67	218	308	207	152	4	956

(ii) Organic Correlations.

TABLE S. *Organic Correlation of Length and Breadth of Egg.*

Length.

Breadth.	3·55—	3·60—	3·65—	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
2·65—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
2·70—	—	—	—	1	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
2·75—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	2
2·80—	—	—	—	1	—	—	1	—	2	—	1	—	—	1	—	—	1	—	1	—	—	—	—	8
2·85—	—	—	—	—	1	2	—	1	—	2	—	3	—	1	4	—	1	—	—	1	—	—	—	16
2·90—	—	—	—	—	3	1	3	3	2	5	9	6	6	3	7	3	3	1	2	—	1	—	—	58
2·95—	—	—	—	—	1	3	3	4	11	10	18	15	12	14	7	8	3	2	3	—	—	—	—	114
3·00—	—	—	—	—	1	5	3	14	14	18	20	30	32	21	17	20	10	9	3	2	6	1	2	228
3·05—	—	—	—	—	—	—	2	2	8	21	19	29	15	24	21	24	10	8	3	3	2	1	1	193
3·10—	—	—	—	—	—	2	2	1	13	6	13	36	23	23	25	26	16	11	6	5	3	—	2	213
3·15—	—	—	—	—	—	—	2	1	3	5	4	6	9	6	7	12	6	5	6	4	—	—	—	76
3·20—	—	—	—	—	—	—	—	2	2	1	1	1	4	4	2	8	4	4	1	—	—	—	—	34
3·25—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	1	5	1	—	—	—	1	—	—	10
Totals	1	0	0	2	6	14	16	30	55	69	85	128	101	98	91	106	55	40	25	15	12	2	5	956

TABLE T. *Organic Correlation of Breadth and Longitudinal Girth of Egg.*

Longitudinal Girth.

Breadth.	9·80—	9·90—	—	10·20—	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
2·65—	—	—	.	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
2·70—	—	—	.	—	1	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
2·75—	1	—	.	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
2·80—	—	—	.	—	1	1	—	1	1	1	—	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	8
2·85—	—	—	.	—	—	—	1	2	—	4	—	2	—	5	—	—	1	1	—	—	—	—	—	—	—	—	16
2·90—	—	—	.	—	—	1	3	4	2	4	6	9	7	3	8	4	5	—	—	2	—	—	—	—	—	—	58
2·95—	—	—	.	—	—	—	1	4	4	13	8	18	17	19	11	6	7	4	1	1	—	—	—	—	—	—	114
3·00—	—	—	.	—	—	—	—	4	2	8	20	23	25	35	31	17	23	14	10	4	8	1	3	—	—	—	228
3·05—	—	—	.	—	—	—	1	—	—	2	3	15	23	31	22	31	18	22	11	8	3	2	1	—	—	—	193
3·10—	—	—	.	—	—	—	—	—	—	2	1	9	11	17	35	31	32	24	23	14	4	5	1	2	1	1	213
3·15—	—	—	.	—	—	—	—	—	—	—	—	1	5	5	6	9	7	7	12	10	8	4	1	1	—	—	76
3·20—	—	—	.	—	—	—	—	—	—	—	—	—	1	4	—	3	3	3	6	8	3	3	—	—	—	—	34
3·25—	—	—	.	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	1	3	1	—	—	—	—	10
Totals	1	0	—	0	2	2	6	15	10	35	38	79	90	119	114	101	98	76	66	48	29	16	6	3	1	1	956

* Character unrecorded.

TABLE U. *Organic Correlation of Length and Index of Egg.*

Index. \ Length.	3·55—	3·60—	3·65—	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
62—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	—	—	3
64—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	3	—	2	1	3	10
66—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	4	3	6	5	5	4	6	1	2	38
68—	—	—	—	—	—	—	—	1	—	1	1	1	1	3	15	16	12	15	8	7	3	—	—	84
70—	—	—	—	—	—	1	—	1	2	2	4	12	22	31	24	40	25	14	8	3	—	—	—	189
72—	—	—	—	1	—	—	1	1	1	10	28	45	36	40	41	34	9	6	—	—	—	—	—	253
74—	—	—	—	1	1	2	3	4	20	28	33	51	32	20	5	13	2	—	—	—	—	—	—	215
76—	1	—	—	—	3	4	4	17	19	21	16	16	10	2	2	—	—	—	—	—	—	—	—	115
78—	—	—	—	—	1	5	4	4	10	7	3	3	—	—	—	—	—	—	—	—	—	—	—	37
80—	—	—	—	—	1	2	3	1	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
82—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	1	0	0	2	6	14	16	30	55	69	85	128	101	98	91	106	55	40	25	15	12	2	5	956

TABLE V. *Organic Correlation of Length and Longitudinal Girth of Egg.*

Longitudinal Girth. \ Length.	3·55—	3·60—	3·65—	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
9·80—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
9·90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10·00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10·10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10·20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10·30—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
10·40—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
10·50—	—	—	—	—	2	2	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	6
10·60—	—	—	—	—	1	6	2	4	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
10·70—	—	—	—	—	—	2	4	—	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	10
10·80—	—	—	—	—	1	3	3	8	9	8	2	1	—	—	—	—	—	—	—	—	—	—	—	35
10·90—	—	—	—	—	1	1	1	11	7	11	5	1	—	—	—	—	—	—	—	—	—	—	—	38
11·00—	—	—	—	1	—	—	1	2	20	20	23	10	1	1	—	—	—	—	—	—	—	—	—	79
11·10—	—	—	—	—	—	—	1	3	6	20	20	25	8	3	1	2	1	—	—	—	—	—	—	90
11·20—	—	—	—	—	—	—	1	1	7	6	21	37	29	11	5	—	—	1	—	—	—	—	—	119
11·30—	—	—	—	—	—	—	—	—	—	—	7	38	25	23	13	5	1	1	1	—	—	—	—	114
11·40—	—	—	—	—	—	—	—	—	—	2	3	10	25	29	21	9	2	—	—	—	—	—	—	101
11·50—	—	—	—	—	—	—	—	—	—	—	1	5	11	20	27	22	9	1	2	—	—	—	—	98
11·60—	—	—	—	—	—	—	—	—	1	—	—	1	2	8	15	30	12	5	1	1	—	—	—	76
11·70—	—	—	—	—	—	—	—	—	—	2	—	—	2	9	25	16	10	2	—	—	—	—	—	66
11·80—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	9	10	15	11	1	1	—	—	—	48
11·90—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	4	2	5	4	6	6	—	—	29
12·00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	4	5	3	1	—	16
12·10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	3	6
12·20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	3
12·30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
Totals	1	0	0	2	6	14	16	30	55	69	85	128	101	98	91	106	55	40	25	15	12	2	5	956

* Character unrecorded.

TABLE W. *Organic Correlation of Longitudinal and Transverse Girths of Egg.*

Longitudinal Girth.

Transverse Girth.	9·80—	9·90—	—	10·20—	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
8·60—	—	—	·	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
8·70—	1	—	·	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
8·80—	—	—	·	—	1	—	—	1	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	4
8·90—	—	—	·	—	—	—	—	—	1	2	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	5
9·00—	—	—	·	—	—	—	1	—	—	3	—	1	—	5	—	—	1	1	—	—	—	—	—	—	—	—	12
9·10—	—	—	·	—	—	—	1	3	—	3	—	4	2	1	3	—	2	—	—	—	—	—	—	—	—	—	19
9·20—	—	—	·	—	—	1	3	2	1	6	7	7	6	1	4	4	2	—	—	1	1	—	—	—	—	—	46
9·30—	—	—	·	—	—	—	1	4	2	6	4	14	13	10	6	1	5	2	1	2	—	—	—	—	—	—	71
9·40—	—	—	·	—	—	—	—	3	3	6	10	12	13	22	14	8	8	3	—	1	1	—	—	—	—	—	104
9·50—	—	—	·	—	—	—	—	2	2	4	11	12	14	22	14	11	13	9	9	4	5	—	3	—	—	—	135
9·60—	—	—	·	—	—	—	—	—	—	3	4	14	15	16	21	22	10	10	5	6	3	2	1	—	—	—	132
9·70—	—	—	·	—	—	—	—	—	—	2	1	9	14	21	24	19	20	18	12	6	2	2	—	—	—	—	150
9·80—	—	—	·	—	—	—	—	—	—	—	—	5	5	9	18	16	19	16	13	7	2	2	1	—	1	—	114
9·90—	—	—	·	—	—	—	—	—	—	—	1	—	5	4	5	7	9	8	9	7	4	4	1	2	—	—	66
10·00—	—	—	·	—	—	—	—	—	—	—	—	1	1	5	4	10	4	5	8	7	6	3	—	1	—	—	55
10·10—	—	—	·	—	—	—	—	—	—	—	—	—	1	2	—	2	3	2	5	6	1	2	—	—	—	—	24
10·20—	—	—	·	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	1	1	2	1	—	—	—	—	7
10·30—	—	—	·	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	2	—	—	—	—	—	5
*	—	—	·	—	—	1	—	—	—	—	—	—	—	—	—	1	—	1	1	—	—	—	—	—	—	1	5
Totals	1	0	—	0	2	2	6	15	10	35	38	79	90	119	114	101	98	76	66	48	29	16	6	3	1	1	956

TABLE X. *Organic Correlation of Longitudinal Girth and Index* (100*B*/*L*) *of Egg.*

Longitudinal Girth.

Index.	9·80—	9·90—	—	10·20—	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
62—	—	—	·	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	—	—	—	—	—	3
64—	—	—	·	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	2	2	—	4	—	—	—	10
66—	—	—	·	—	—	—	—	—	—	—	—	—	1	5	2	3	4	4	2	5	7	3	—	1	1	—	38
68—	—	—	·	—	—	—	—	—	1	1	—	2	3	3	12	7	15	9	10	9	4	5	2	1	—	—	84
70—	—	—	·	—	1	—	—	1	1	4	1	10	10	21	23	20	25	26	20	14	5	5	—	1	—	1	189
72—	—	—	·	—	—	1	—	1	—	7	7	23	26	35	28	31	30	27	19	11	5	2	—	—	—	—	253
74—	—	—	·	—	1	—	3	5	4	9	12	21	24	30	34	27	15	6	12	5	6	1	—	—	—	—	215
76—	1	—	·	—	—	1	2	4	2	9	15	15	19	14	13	9	5	2	3	1	—	—	—	—	—	—	115
78—	—	—	·	—	—	—	1	3	2	3	2	7	4	7	1	4	2	1	—	—	—	—	—	—	—	—	37
80—	—	—	·	—	—	—	—	1	—	2	1	1	3	2	—	—	—	—	—	—	—	—	—	—	—	—	10
82—	—	—	·	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	1	0	—	0	2	2	6	15	10	35	38	79	90	119	114	101	98	76	66	48	29	16	6	3	1	1	956

TABLE Y. *Organic Correlation of Breadth and Index* (100*B*/*L*) *of Egg.*

Breadth.

Index.	2·65—	2·70—	2·75—	2·80—	2·85—	2·90—	2·95—	3·00—	3·05—	3·10—	3·15—	3·20—	3·25—	Totals
62—	—	—	—	1	1	1	—	—	—	—	—	—	—	3
64—	—	—	—	1	—	2	1	5	1	—	—	—	—	10
66—	—	—	—	1	6	7	6	10	6	2	—	—	—	38
68—	—	1	1	1	1	11	15	28	12	13	1	—	—	84
70—	1	1	—	2	4	15	30	50	41	32	12	1	—	189
72—	—	1	—	1	1	11	35	64	51	59	22	8	—	253
74—	—	—	—	1	3	7	17	42	51	62	14	12	6	215
76—	—	—	1	—	—	4	9	21	26	32	14	6	2	115
78—	—	—	—	—	—	—	1	7	5	9	10	3	2	37
80—	—	—	—	—	—	—	—	1	—	4	2	3	—	10
82—	—	—	—	—	—	—	—	—	—	—	1	1	—	2
Totals	1	3	2	8	16	58	114	228	193	213	76	34	10	956

* Character unrecorded.

TABLE Z. *Organic Correlation of Mottling and Breadth of Egg.*

Mottling.

Breadth.	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*	Totals
2·65—	—	—	—	—	—	1	—	—	—	—	—	—	1
2·70—	·5	—	—	—	1·5	—	1	—	—	—	—	—	3
2·75—	1	—	—	—	—	—	1	—	—	—	—	—	2
2·80—	·5	1	1	—	1·5	—	1	—	—	1	—	2	8
2·85—	5	2	1	2	2·5	1·5	1	—	—	1	—	—	16
2·90—	7·5	10·5	17	3	5·5	5	1	—	3·5	4	—	1	58
2·95—	25·5	17	14·5	7	11·5	15·5	7·5	6	6·5	3	—	—	114
3·00—	61·5	31	36·5	8	27	23·5	13	9·5	8·5	5·5	3	1	228
3·05—	44	23	31·5	5	37	21	15·5	2·5	8·5	2	2	1	193
3·10—	54·5	26·5	17·5	8	40·5	32·5	12·5	6	8	4·5	1·5	1	213
3·15—	20·5	7·5	6	2	16·5	14	2·5	5	1	1	—	—	76
3·20—	12	2·5	2·5	2	7	3	2	1	1·5	·5	—	—	34
3·25—	1	1·5	1	1	2·5	1	—	—	1	—	1	—	10
Totals	233·5	122·5	128·5	38	153	118	58	30	38·5	22·5	7·5	6	956

TABLE AA. *Organic Correlation of Ground-Colour and Breadth of Egg.*

Categories of Ground-Colour.

Breadth.	*2*	*3*	*4*	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	D_1	D_2	D_3	D_4	*	Totals
2·65—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
2·70—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	—	—	—	—	—	—	3
2·75—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2
2·80—	—	—	1	—	—	—	·5	—	—	1	1·5	—	—	1	1	—	—	—	—	2	8
2·85—	—	—	1	—	—	1	—	—	—	4	—	—	2	3	1	—	—	4	—	—	16
2·90—	1	3	4	—	2	7	—	—	4·5	4	2	—	7	7	2	—	2	11·5	1	—	58
2·95—	2	5	2	—	1	12	1	—	4	18	1	—	8	22	4	—	3·5	22·5	8	—	114
3·00—	1·5	6	9	—	8	25	4	2	9	40	4	1	19	32·5	8·5	1	13·5	30·5	11·5	2	228
3·05—	3	9	6	—	6	19·5	12·5	1·5	10	24	6	1	11	25·5	6·5	1	9	28·5	13	—	193
3·10—	—	11	8	1	7	21·5	3·5	4	8	24	8	3	22	28	9	1	8	27	16	3	213
3·15—	2	2	2	—	4	8	2	1	3	9	3	2	5	9	2	—	1	17	3	1	76
3·20—	1	—	1	—	3	—	—	—	—	6	1	—	3	6	2	—	2	5	3	1	34
3·25—	—	—	—	—	1	—	—	—	—	3	1	—	1	—	1	—	—	3	—	—	10
Totals	10·5	36	35	1	32	94	23·5	8·5	38·5	135	27·5	7	79	135	37	3	39	149	56·5	9	956

TABLE BB. *Organic Correlation of Mottling and Length of Egg.*

Length.

Mottling.	*3·55*—	*3·60*—	*3·65*—	*3·70*—	*3·75*—	*3·80*—	*3·85*—	*3·90*—	*3·95*—	*4·00*—	*4·05*—	*4·10*—	*4·15*—	*4·20*—	*4·25*—	*4·30*—	*4·35*—	*4·40*—	*4·45*—	*4·50*—	*4·55*—	*4·60*—	*4·65*—	Totals
a	1	—	—	—	1	2·5	4·5	7	16	17	12	31·5	26	28·5	25·5	22·5	15·5	13	4	3	2	—	1	233·5
b	—	—	—	1	1	1	3	7	11·5	6	11·5	21	15·5	14	9	9	5·5	3	1	—	2·5	—	—	122·5
c	—	—	—	—	·5	3	3	3·5	5	11	10	10·5	14	8·5	19	17·5	6	5	6	3	2	—	1	128·5
d	—	—	—	—	2	—	—	—	—	3·5	4	4·5	4	7	2	6	2	1	1	1	—	—	—	38
e	—	—	—	1	·5	2·5	2	3	3·5	13	18	16	13·5	12·5	19	20·5	9·5	9	6	·5	1	1	1	153
f	—	—	—	—	—	2	1	—	10	9	13	17	11	15·5	5·5	13	7	3	2	6·5	2·5	—	—	118
g	—	—	—	—	—	—	—	4	3	3	7·5	14·5	4	7	2	5	1·5	3	1·5	—	1	—	1	58
h	—	—	—	—	—	—	—	—	2	2·5	3·5	2·5	4	1	3·5	4	2	1	1	1	1	—	1	30
i	—	—	—	—	·5	2	1·5	2	3	4	3	5	5	—	1	3	4·5	2	2	—	—	—	—	38·5
j	—	—	—	—	·5	1	—	3	1	—	1·5	2	2	2	2	5·5	1·5	—	·5	—	—	—	—	22·5
k	—	—	—	—	—	—	—	·5	—	—	—	2·5	2	1	·5	—	—	—	—	—	—	1	—	7·5
*	—	—	—	—	—	—	1	—	—	—	1	1	—	1	2	—	—	—	—	—	—	—	—	6
Totals	1	0	0	2	6	14	16	30	55	69	85	128	101	98	91	106	55	40	25	15	12	2	5	956

* Character unrecorded.

TABLE CC. *Organic Correlation of Mottling and Longitudinal Girth of Egg.*

Longitudinal Girth.

Mottling.	9·80—	9·90—	—	10·20—	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
a	1	—	.	—	·5	—	—	5·5	1	10·5	4·5	18	19	33	38	25	24·5	13	17·5	11·5	8	2	1	—	—	—	233·5
b	—	—	.	—	1	—	1	2	1	6	7·5	17	11	15·5	11·5	16	13	12	1	4·5	1·5	—	1	—	—	—	122·5
c	—	—	.	—	—	·5	2	4	2	3	5·5	8	16	12	11	12·5	21	11	7	5	5	2	1	—	—	—	128·5
d	—	—	.	—	—	—	—	—	—	1	2	2	4·5	8·5	2	6	3	2	4	1	1	1	—	—	—	—	38
e	—	—	.	—	·5	—	·5	1·5	—	3	5·5	16	10	17	16	17·5	14·5	8·5	22	12	4	1	·5	1	1	1	153
f	—	—	.	—	—	—	—	1	3	3	5	6	13	14	14	10	7	19·5	4	6·5	5·5	5	·5	1	—	—	118
g	—	—	.	—	—	—	—	—	1	1	4	7	4	8	10·5	6·5	5·5	1	4·5	2·5	—	1·5	—	1	—	—	58
h	—	—	.	—	—	—	—	—	—	2	1	2	1·5	4·5	3	1	3	1	4	2	3	1	1	—	—	—	30
i	—	—	.	—	—	2·5	—	1	2·5	2·5	3	6·5	2·5	3·5	1	2	5·5	2	1	1	2	—	—	—	—	—	38·5
j	—	—	.	—	·5	—	1	1	3	—	—	1·5	3	3	3·5	1·5	2	—	2	—	·5	—	—	—	—	—	22·5
k	—	—	.	—	—	—	—	—	—	·5	—	1	1	·5	—	3	·5	—	—	—	—	1	—	—	—	—	7·5
*	—	—	.	—	—	1	—	—	—	—	—	2	—	1	2	—	—	—	—	—	—	—	—	—	—	—	6
Totals	1	0	—	0	2	2	6	15	10	35	38	79	90	119	114	101	98	76	66	48	29	16	6	3	1	1	956

TABLE DD. *Organic Correlation of Mottling and Index* (100*B*/*L*) *of Egg.*

Mottling.

Index.	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*	Totals
62—	1	—	1	—	—	—	1	—	—	—	—	—	3
64—	1·5	·5	3	—	1·5	·5	—	2	—	—	1	—	10
66—	11	3	5	4	2·5	6·5	2	—	2	1	—	1	38
68—	18·5	8	18	1	13·5	9·5	7·5	3	2	3	—	—	84
70—	43·5	27	34	8	30·5	14·5	10	6·5	5·5	8	·5	1	189
72—	63·5	25	26	13	48	41	9·5	7	11	4	2	3	253
74—	47	33	24	8·5	28·5	31	21	10	7	2	2	1	215
76—	33	16·5	13·5	2·5	21	11	4·5	1·5	7	3·5	1	—	115
78—	11	7·5	3	1	5	4	1·5	—	3	—	1	—	37
80—	1·5	2	1	—	2·5	—	1	—	1	1	—	—	10
82—	2	—	—	—	—	—	—	—	—	—	—	—	2
Totals	233·5	122·5	128·5	38	153	118	58	30	38·5	22·5	7·5	6	956

TABLE EE. *Organic Correlation of Ground-Colour and Length of Egg.*

Length.

Ground-Colour.	3·55—	3·60—	3·65—	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
2	—	—	—	—	—	—	—	—	—	—	3	1	·5	1	1	1	—	2	—	1	—	—	—	10·5
3	—	—	—	—	—	1	—	1	4	—	2	7	6	1	3	4	4	1	1	2	—	—	—	36
4	—	—	—	—	—	—	1	—	1	3	6	8	3	—	2	5	2	2	—	2	—	—	—	35
A_1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
A_2	—	—	—	—	—	1	2	1	2	1	3	2	3	1	2	5	1	3	1	1	3	—	—	32
A_3	—	—	—	—	2	3	1	4	—	10·5	6	12·5	12	5	11	9	9	3	2	—	1	—	—	94
A_4	—	—	—	·5	—	—	—	1	2	·5	3	7·5	1	1	2	3	—	1	—	1	—	—	—	23·5
B_1	—	—	—	—	—	—	·5	—	—	—	1	—	—	4	1	—	—	1	—	1	—	—	—	8·5
B_2	—	—	—	—	—	1	—	2	5	3	6	7·5	3	2	6	—	1	1	—	1	—	—	—	38·5
B_3	—	—	—	—	1	2	2·5	4	10	9	8	18	8·5	18	11	21	6	8·5	5	—	·5	—	2	135
B_4	—	—	—	·5	—	—	—	—	1	1	3	1	1	7	3	1	3	1	3	1	1	—	—	27·5
C_1	—	—	—	—	—	—	—	—	1	—	2	1	—	—	—	1	—	—	1	1	—	—	—	7
C_2	—	—	—	—	—	1	—	4	7	3	4	10·5	3·5	9	16	8	5	4	—	—	2	—	2	79
C_3	—	—	—	1	—	3	1	7	5	15	13·5	20·5	18·5	14	14	10	7	2·5	1	2	—	—	—	135
C_4	—	—	—	—	1	—	—	2	3·5	1	5	1·5	5	—	6	2	—	1	2	—	2	—	—	37
D_1	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	3
D_2	—	—	—	—	—	1	2	2	—	4·5	2·5	7	2	2	3	5	1	1	3	1	1	1	—	39
D_3	—	—	—	—	1	2	1	4	10	12·5	15	15·5	19·5	23	10	17	8	4	3	2	·5	1	—	149
D_4	1	—	—	—	1	—	2	2	1·5	3	5	5	9	8	3	7	2	4	2	—	—	—	1	56·5
*	—	—	—	—	—	—	1	—	1	—	—	3	1	1	1	1	—	—	—	—	—	—	—	9
Totals	1	0	0	2	6	14	16	30	55	69	85	128	101	98	91	106	55	40	25	15	12	2	5	956

* Character unrecorded.

TABLE FF. *Organic Correlation of Ground-Colour and Index (100B/L) of Egg.*

Index. \ Ground-Colour.	2	3	4	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	D_1	D_2	D_3	D_4	*	Totals
62—	—	—	1	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	3
64—	—	1	—	—	—	1	—	—	—	3	—	—	1	—	2	—	1	—	1	—	10
66—	—	—	1	—	6	4	1	1	2	5·5	1	—	4	1	1	—	2	6·5	1	1	38
68—	2	5	5	—	2	10	2	—	2·5	15·5	4	2	3·5	8·5	3	—	5	12	2	—	84
70—	2	10	6	1	2	16	2	2	6	30	7	1	22·5	27	6	1	8	30·5	8	1	189
72—	4·5	6	8	—	7	23	7	3	13	30·5	8	—	23	39·5	9	1	6·5	38	23	3	253
74—	1	9	11	—	7	21·5	7	1	10	29	5·5	2	15	34	5·5	—	7·5	37	10	2	215
76—	1	3	3	—	5	7·5	4·5	—	5	16	2	1	5	18	7·5	1	6	19	9·5	1	115
78—	—	2	—	—	—	9	—	1·5	—	5·5	—	—	3	5	2	—	2	6	1	—	37
80—	—	—	—	—	2	2	—	—	—	—	—	1	1	1	—	—	1	—	1	1	10
82—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	2
Totals	10·5	36	35	1	32	94	23·5	8·5	38·5	135	27·5	7	79	135	37	3	39	149	56·5	9	956

TABLE GG. *Organic Correlation of Ground-Colour and Mottling of Egg.*

Mottling. \ Ground-Colour.	2	3	4	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	D_1	D_2	D_3	D_4	*	Totals
a	—	10·5	5·5	—	7·5	20	8	4	10	31·5	10	—	23	39	12·5	1	7	28·5	15	·5	233·5
b	1	6	4·5	—	2	18	4	1·5	3	19·5	3·5	2	11·5	15·5	4	1	3·5	17·5	4·5	—	122·5
c	1	6	6·5	—	2	10	2·5	2	4	23	3	—	8·5	18·5	3·5	—	7	24·5	4·5	2	128·5
d	1·5	—	1	—	3	2	—	1	—	3	1	—	5	7	2	—	1	5·5	5	—	38
e	2·5	4	6	1	4·5	13	4·5	—	5·5	19·5	5	1	12	19	5	—	9	27·5	13	1	153
f	—	7	5	—	7	13	3	—	10	15·5	—	1	5·5	15·5	1·5	—	8	18	7·5	·5	118
g	2	2	2	—	3	7·5	1·5	—	2	9·5	1	1	5	4·5	2	—	2	11·5	·5	1	58
h	1	—	1	—	1	4	—	—	2	4	1	2	2	3·5	2·5	—	—	4	2	—	30
i	1·5	—	·5	—	1	5	—	—	1·5	7	2	—	6	4·5	1	—	—	4·5	2·5	1·5	38·5
j	—	·5	3	—	1	1	—	—	·5	2	1	—	·5	4	2	1	—	3·5	2	·5	22·5
k	—	—	—	—	—	·5	—	—	—	·5	—	—	—	2	—	—	1·5	3	—	—	7·5
*	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	—	1	—	2	6
Totals	10·5	36	35	1	32	94	23·5	8·5	38·5	135	27·5	7	79	135	37	3	39	149	56·5	9	956

(iii) Direct Homotyposis of Eggs of same Clutch.

TABLE HH. *Direct Homotyposis of Longitudinal Girths.*

Longitudinal Girth of Second Egg. \ Longitudinal Girth of First Egg.	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
10·30—	—	—	2	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
10·40—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	2
10·50—	2	—	—	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
10·60—	—	—	2	6	5	4	3	3	1	—	1	—	—	1	1	—	—	—	—	—	—	—	27
10·70—	—	—	—	5	—	4	2	3	1	1	—	—	1	—	—	—	—	—	—	—	—	—	17
10·80—	1	—	2	4	4	4	9	12	4	3	2	2	1	—	—	1	—	—	—	—	—	—	49
10·90—	—	—	—	3	2	9	6	7	11	4	3	4	1	2	1	1	—	—	—	—	—	—	54
11·00—	—	—	—	3	3	12	7	30	19	21	13	4	3	2	—	—	—	—	—	—	—	—	117
11·10—	—	—	—	1	1	4	11	19	24	30	20	14	7	3	3	2	2	—	—	—	—	—	141
11·20—	—	1	—	—	1	3	4	21	30	36	37	25	19	7	5	1	—	—	—	1	—	—	191
11·30—	—	1	—	1	—	2	3	13	20	37	24	28	31	11	11	1	2	—	1	—	—	—	186
11·40—	—	—	—	—	—	2	4	4	14	25	28	28	16	14	12	7	1	—	2	—	—	1	158
11·50—	—	—	—	—	1	1	1	3	7	19	31	16	28	26	15	10	6	—	—	—	—	—	164
11·60—	—	—	—	1	—	—	2	2	3	7	11	14	26	26	14	16	3	3	—	—	—	—	128
11·70—	—	—	—	1	—	—	1	—	3	5	11	12	15	14	20	15	5	5	1	1	—	—	109
11·80—	—	—	—	—	—	1	1	—	2	1	1	7	10	16	15	22	10	2	—	—	—	—	88
11·90—	—	—	—	—	—	—	—	—	2	—	2	1	6	3	5	10	10	5	2	2	1	1	50
12·00—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	5	2	5	2	3	—	1	—	21
12·10—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	1	—	2	3	2	—	—	—	11
12·20—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	2	—	—	2	—	—	6
12·30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	2
*	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	2
Totals	3	2	6	27	17	49	54	117	141	191	186	158	164	128	109	88	50	21	11	6	2	2	1532

* Character unrecorded.

TABLE II. *Direct Homotyposis of Transverse Girths.*

Transverse Girth of First Egg.

Transverse Girth of Second Egg.	8·60—	8·70—	8·80—	8·90—	9·00—	9·10—	9·20—	9·30—	9·40—	9·50—	9·60—	9·70—	9·80—	9·90—	10·00—	10·10—	10·20—	10·30—	*	Totals
8·60—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	2
8·70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
8·80—	—	—	—	1	2	—	—	—	—	—	—	—	—	1	—	—	—	—	—	4
8·90—	—	—	1	—	2	—	—	2	1	—	—	—	—	1	—	—	—	—	—	7
9·00—	—	—	2	2	—	1	5	5	1	—	—	—	—	—	—	—	—	—	—	16
9·10—	—	—	—	—	1	2	5	2	6	1	2	3	—	1	—	—	—	—	—	23
9·20—	—	—	—	—	5	5	12	15	13	8	5	5	1	—	—	—	—	—	1	70
9·30—	1	—	—	2	5	2	15	6	23	21	19	12	5	3	—	—	—	—	—	114
9·40—	—	—	—	1	1	6	13	23	44	25	28	13	4	3	1	—	—	—	—	162
9·50—	—	—	—	—	—	1	8	21	25	48	32	46	24	9	5	1	—	—	2	222
9·60—	1	—	—	—	—	2	5	19	28	32	54	32	23	15	7	3	—	—	—	221
9·70—	—	—	—	—	—	3	5	12	13	46	32	42	46	21	14	4	1	—	2	241
9·80—	—	—	—	—	—	—	1	5	4	24	23	46	32	22	17	7	—	—	1	182
9·90—	—	—	1	1	—	1	—	3	3	9	15	21	22	12	14	6	1	2	1	112
10·00—	—	—	—	—	—	—	—	—	1	5	7	14	17	14	22	6	2	2	2	92
10·10—	—	—	—	—	—	—	—	—	—	1	3	4	7	6	6	4	2	2	—	35
10·20—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2	2	2	3	—	11
10·30—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	2	3	—	—	9
*	—	—	—	—	—	—	1	—	—	2	—	2	1	1	2	—	—	—	—	9
Totals	2	0	4	7	16	23	70	114	162	222	221	241	182	112	92	35	11	9	9	1532

TABLE JJ. *Direct Homotyposis of Lengths.*

Length of First Egg.

Length of Second Egg.	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
3·70—	—	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
3·75—	—	—	3	2	3	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	10
3·80—	2	3	2	4	5	4	1	2	—	—	—	—	—	—	—	—	—	—	—	1	23
3·85—	—	2	4	2	3	4	2	1	2	1	—	—	—	—	—	—	—	—	—	—	21
3·90—	1	3	5	3	2	11	6	4	6	3	3	1	2	1	—	—	—	—	—	—	51
3·95—	—	—	4	4	11	18	21	6	11	1	4	4	2	1	—	—	—	—	—	—	87
4·00—	—	—	1	2	6	21	20	7	18	8	11	7	3	—	—	—	—	—	—	—	104
4·05—	—	—	2	1	4	6	7	18	33	21	15	4	5	1	1	1	1	—	—	—	120
4·10—	—	—	—	2	6	11	18	33	34	33	25	13	18	3	4	1	1	1	—	—	203
4·15—	—	1	—	1	3	1	8	21	33	36	17	14	18	7	4	2	—	—	—	—	166
4·20—	—	—	—	—	3	4	11	15	25	17	22	27	19	7	4	4	3	—	1	2	164
4·25—	—	—	—	—	1	4	7	4	13	14	27	20	19	16	17	4	1	1	—	—	148
4·30—	—	—	—	—	2	2	3	5	18	18	19	19	46	19	13	11	2	1	—	—	178
4·35—	—	1	—	—	1	1	—	1	3	7	7	16	19	12	8	3	2	5	1	—	87
4·40—	—	—	—	—	—	—	—	1	4	4	4	17	13	8	8	5	1	1	—	1	67
4·45—	—	—	—	—	—	—	—	1	1	2	4	4	11	3	5	2	5	4	—	3	45
4·50—	—	—	—	—	—	—	—	1	1	—	3	1	2	2	1	5	2	3	—	2	23
4·55—	—	—	—	—	—	—	—	—	1	—	—	1	1	5	1	4	3	4	—	—	20
4·60—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	1	3
4·65—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	1	3	2	—	1	—	9
Totals	3	10	23	21	51	87	104	120	203	166	164	148	178	87	67	45	23	20	3	9	1532

* Character unrecorded.

TABLE KK. *Direct Homotyposis of Indices* (100*B*/*L*).

Index of Second Egg.	Index of First Egg. 62—	64—	66—	68—	70—	72—	74—	76—	78—	80—	82—	Totals
62—	—	—	2	2	—	—	—	—	—	—	—	4
64—	—	—	1	5	7	3	—	—	—	—	—	16
66—	2	1	8	14	19	12	. 4	—	—	—	—	60
68—	2	5	14	34	37	36	10	3	—	—	—	141
70—	—	7	19	37	68	98	44	18	6	1	—	298
72—	—	3	12	36	98	140	95	33	6	1	1	425
74—	—	—	4	10	44	95	90	62	22	2	—	329
76—	—	—	—	3	18	33	62	52	12	3	1	184
78—	—	—	—	—	6	6	22	12	10	4	—	60
80—	—	—	—	—	1	1	2	3	4	2	—	13
82—	—	—	—	—	—	1	—	1	—	—	—	2
Totals	4	16	60	141	298	425	329	184	60	13	2	1532

TABLE LL. *Direct Homotyposis of Breadths.*

Breadth of Second Egg.	Breadth of First Egg. 2·65—	2·70—	2·75—	2·80—	2·85—	2·90—	2·95—	3·00—	3·05—	3·10—	3·15—	3·20—	3·25—	Totals
2·65—	—	2	—	—	—	—	—	—	—	—	—	—	—	2
2·70—	2	2	—	—	—	—	1	1	—	—	—	—	—	6
2·75—	—	—	—	1	—	—	—	—	—	1	—	—	—	2
2·80—	—	—	1	—	3	2	1	1	—	2	—	—	—	10
2·85—	—	—	—	3	—	7	6	1	—	1	—	—	—	18
2·90—	—	—	—	2	7	20	22	25	9	2	—	—	—	87
2·95—	—	1	—	1	6	22	38	69	29	13	2	—	—	181
3·00—	—	1	—	1	1	25	69	104	89	70	10	5	—	375
3·05—	—	—	—	—	—	9	29	89	72	80	22	6	—	307
3·10—	—	—	1	2	1	2	13	70	80	112	51	14	2	348
3·15—	—	—	—	—	—	—	2	10	22	51	22	14	5	126
3·20—	—	—	—	—	—	—	—	5	6	14	14	6	7	52
3·25—	—	—	—	—	—	—	—	—	—	2	5	7	4	18
Totals	2	6	2	10	18	87	181	375	307	348	126	52	18	1532

TABLE MM. *Direct Homotyposis of Mottlings.*

Mottling of Second Egg.	Mottling of First Egg. *a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*	Totals
a	188	57·75	45·25	18	26·75	20	8·5	2	9·25	3·5	·5	6·5	386
b	57·75	34	46·75	9	17	11·5	5·5	5·25	4·75	2·5	—	·5	194·5
c	45·25	46·75	38	9	15	19	5·5	3	2·5	4	—	—	188
d	18	9	9	6·5	3·5	3	3	1·5	1·5	1	—	—	56
e	26·75	17	15	3·5	95	34·5	23	14·5	13	7·25	2·5	3	255
f	20	11·5	19	3	34·5	52	22·5	7·5	8·5	9	4·5	—	192
g	8·5	5·5	5·5	3	23	22·5	13·5	4	3·5	5	·5	—	94·5
h	2	5·25	3	1·5	14·5	7·5	4	8·5	3	1·75	—	—	51
i	9·25	4·75	2·5	1·5	13	8·5	3·5	3	8	1	1·5	—	56·5
j	3·5	2·5	4	1	7·25	9	5	1·75	1	1	·5	—	36·5
k	·5	—	—	—	2·5	4·5	·5	—	1·5	·5	2	—	12
*	6·5	·5	—	—	3	—	—	—	—	—	—	—	10
Totals	386	194·5	188	56	255	192	94·5	51	56·5	36·5	12	10	1532

* Character unrecorded.

TABLE NN. *Direct Homotyposis of Ground-Colours.*

Ground-Colour of First Egg.

Ground-Colour of Second Egg.	*2*	*3*	*4*	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	D_1	D_2	D_3	D_4	*	Totals
2	3	2	—	—	6	1	—	—	1	1·75	—	—	—	1·5	—	—	—	·75	—	1	18
3	2	20	8	—	1	14	—	—	2	5	—	—	—	4	—	—	—	—	—	2	58
4	—	8	28	—	—	6·5	6·5	—	—	2	1	—	2	1	3	—	—	1	—	—	59
A_1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
A_2	6	1	—	—	12	8	—	1	5	8	—	—	2	2	—	—	—	6	—	—	51
A_3	1	14	6·5	—	8	66	1·5	—	8	29	2·5	—	4	13	1	—	2	3	—	—	159·5
A_4	—	—	6·5	—	—	1·5	11	—	—	2	9·5	—	—	2·5	3	—	—	—	—	—	36
B_1	—	—	—	—	1	—	—	8	3	—	—	—	4	—	—	—	—	—	—	—	16
B_2	1	2	—	—	5	8	—	3	10	7	—	—	12	5	—	—	5	2	2	—	62
B_3	1·75	5	2	—	8	29	2	—	7	78	4	—	7	48·25	6	—	—	14	6	1	219
B_4	—	—	1	—	—	2·5	9·5	—	—	4	8	—	—	5·5	10	—	—	2	4	—	46·5
C_1	—	—	—	—	—	—	—	—	—	—	—	2	4	—	—	—	—	2	—	—	8
C_2	—	—	2	—	2	4	—	4	12	7	—	4	46	15	—	—	16	17	2	2	133
C_3	1·5	4	1	—	2	13	2·5	—	5	48·25	5·5	—	15	45	9·5	—	5·75	42	9	—	209
C_4	—	—	3	—	—	1	3	—	—	6	10	—	—	9·5	16	—	·25	4·25	8	—	61
D_1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2
D_2	—	—	—	—	—	2	—	—	5	—	—	—	16	5·75	·25	—	26	12	—	—	67
D_3	·75	—	1	—	6	3	—	—	2	14	2	2	17	42	4·25	1	12	102	20	2	231
D_4	—	—	—	—	—	—	—	—	2	6	4	—	2	9	8	1	—	20	30	—	82
*	1	2	—	—	—	—	—	—	—	1	—	—	2	—	—	—	—	2	—	6	14
Totals	18	58	59	0	51	159·5	36	16	62	219	46·5	8	133	209	61	2	67	231	82	14	1532

* Character unrecorded.

(iv) Cross-Homotyposis of Eggs of same Clutch.

TABLE OO. *Cross-Homotyposis of Longitudinal and Transverse Girths.*

Longitudinal Girth of First Egg.

Transverse Girth of Second Egg.	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
8·60—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
8·70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
8·80—	—	—	1	—	—	1	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	4
8·90—	—	—	—	—	1	1	—	1	—	2	1	—	1	—	—	—	—	—	—	—	—	—	7
9·00—	1	—	—	—	1	1	—	2	2	3	1	3	2	—	—	—	—	—	—	—	—	—	16
9·10—	—	—	—	2	2	—	2	4	1	2	3	4	1	2	—	—	—	—	—	—	—	—	23
9·20—	—	—	1	5	1	7	6	11	7	9	13	1	7	1	1	—	—	—	—	—	—	—	70
9·30—	1	—	1	4	3	8	7	14	18	14	10	6	11	7	3	4	2	—	—	1	—	—	114
9·40—	—	—	1	9	4	8	8	17	20	23	21	15	13	6	7	6	2	2	—	—	—	—	162
9·50—	—	1	—	3	3	8	10	18	27	37	33	24	22	14	4	8	7	1	2	—	—	—	222
9·60—	1	—	1	1	—	8	7	23	25	29	26	28	23	23	11	10	2	2	1	—	—	—	221
9·70—	—	1	—	2	1	3	7	15	18	37	37	24	24	26	19	15	6	2	3	1	—	—	241
9·80—	—	—	—	—	—	1	2	6	11	18	23	23	30	20	20	12	11	3	2	—	—	—	182
9·90—	—	—	—	1	1	2	2	5	5	9	8	13	13	14	11	10	8	5	3	1	—	1	112
10·00—	—	—	—	—	—	—	2	—	5	6	6	10	11	9	17	15	4	2	—	3	1	1	92
10·10—	—	—	—	—	—	—	1	—	2	—	2	5	2	3	11	3	3	2	—	—	1	—	35
10·20—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	2	2	3	1	—	—	—	—	11
10·30—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	2	1	1	—	—	—	—	9
*	—	—	—	—	—	—	—	1	—	1	2	2	1	—	—	1	1	—	—	—	—	—	9
Totals	3	2	6	27	17	49	54	117	141	191	186	158	164	128	109	88	50	21	11	6	2	2	1532

TABLE PP. *Cross-Homotyposis of Length and Longitudinal Girth.*

Length of First Egg.

Longitudinal Girth of Second Egg.	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
10·30—	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
10·40—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	2
10·50—	1	—	—	2	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
10·60—	—	3	6	3	5	4	1	2	1	—	—	—	1	1	—	—	—	—	—	—	27
10·70—	—	1	2	2	1	2	2	4	1	2	—	—	—	—	—	—	—	—	—	—	17
10·80—	1	3	3	5	4	9	11	5	2	2	2	1	—	1	—	—	—	—	—	—	49
10·90—	—	—	3	4	7	11	11	—	8	4	1	2	2	1	—	—	—	—	—	—	54
11·00—	1	—	4	—	8	18	18	17	22	15	7	2	3	—	—	2	—	—	—	—	117
11·10—	—	1	2	1	6	14	16	19	25	14	21	7	6	4	3	—	2	—	—	—	141
11·20—	—	—	—	2	6	8	11	32	39	28	28	17	10	5	2	2	—	1	—	—	191
11·30—	—	—	1	1	5	8	14	19	28	32	19	22	19	8	4	4	—	1	—	1	186
11·40—	—	1	—	—	3	6	8	8	23	24	21	23	19	10	5	3	2	—	1	1	158
11·50—	—	—	—	—	—	1	10	8	24	19	25	26	21	7	13	5	3	2	—	—	164
11·60—	—	—	—	—	3	1	2	2	13	12	16	13	37	8	12	7	—	2	—	—	128
11·70—	—	—	—	—	1	3	—	3	11	11	9	16	20	21	7	2	2	2	1	—	109
11·80—	—	—	1	—	—	1	—	—	1	—	8	14	26	11	12	8	2	4	—	—	88
11·90—	—	—	—	—	—	—	—	1	2	1	3	5	9	3	6	5	7	6	—	2	50
12·00—	—	—	—	—	—	—	—	—	1	—	—	—	5	5	2	1	2	2	—	3	21
12·10—	—	—	—	—	—	—	—	—	—	—	3	—	—	1	—	3	2	—	1	1	11
12·20—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	2	1	—	—	1	6
12·30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	2
*	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	2
Totals	3	10	23	21	51	87	104	120	203	166	164	148	178	87	67	45	23	20	3	9	1532

* Character unrecorded.

TABLE QQ. *Cross-Homotyposis of Breadth and Longitudinal Girth.*

Longitudinal Girth of First Egg.

Breadth of Second Egg.	10·30—	10·40—	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	*	Totals
2·65—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
2·70—	—	—	1	—	—	3	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
2·75—	—	—	—	—	—	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	2
2·80—	—	—	1	—	1	—	—	1	—	4	2	—	1	—	—	—	—	—	—	—	—	—	10
2·85—	1	—	—	1	—	1	1	4	2	2	1	3	2	—	—	—	—	—	—	—	—	—	18
2·90—	—	—	1	7	4	7	6	11	8	8	12	6	10	5	1	1	—	—	—	—	—	—	87
2·95—	1	—	1	7	5	9	10	23	25	25	20	12	16	10	6	6	2	2	—	1	—	—	181
3·00—	1	1	2	8	5	17	17	32	45	60	51	43	38	23	9	11	8	2	2	—	—	—	375
3·05—	—	—	—	2	—	8	10	26	27	44	50	33	25	29	19	22	5	2	4	1	—	—	307
3·10—	—	1	—	2	2	3	6	15	25	37	38	40	50	42	37	21	20	5	3	1	—	—	348
3·15—	—	—	—	—	—	—	2	1	7	10	9	13	16	9	20	17	8	6	2	3	1	2	126
3·20—	—	—	—	—	—	—	2	—	2	1	3	8	3	7	12	7	4	2	—	—	1	—	52
3·25—	—	—	—	—	—	—	—	—	—	—	—	—	2	3	5	3	3	2	—	—	—	—	18
Totals	3	2	6	27	17	49	54	117	141	191	186	158	164	128	109	88	50	21	11	6	2	2	1532

TABLE RR. *Cross-Homotyposis of Length and Breadth.*

Length of First Egg.

Breadth of Second Egg.	3·70—	3·75—	3·80—	3·85—	3·90—	3·95—	4·00—	4·05—	4·10—	4·15—	4·20—	4·25—	4·30—	4·35—	4·40—	4·45—	4·50—	4·55—	4·60—	4·65—	Totals
2·65—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
2·70—	1	1	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
2·75—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	2
2·80—	—	—	1	—	—	—	1	—	2	3	2	1	—	—	—	—	—	—	—	—	10
2·85—	1	—	1	—	1	1	1	2	1	2	3	2	—	2	—	1	—	—	—	—	18
2·90—	—	2	4	5	3	7	7	10	7	8	7	6	11	4	2	3	1	—	—	—	87
2·95—	—	3	4	6	8	11	14	25	23	18	16	21	11	9	3	3	2	4	—	—	181
3·00—	—	4	7	6	14	23	38	28	59	51	43	35	28	13	6	11	4	4	—	1	375
3·05—	—	—	1	—	11	25	15	25	45	32	41	24	31	16	20	12	1	4	2	2	307
3·10—	—	—	2	3	10	16	19	24	46	35	34	40	51	25	20	8	7	6	1	1	348
3·15—	—	—	1	—	2	3	7	3	13	10	13	12	26	8	9	5	8	2	—	4	126
3·20—	—	—	—	—	—	1	2	2	6	5	3	5	12	8	5	2	—	—	—	1	52
3·25—	—	—	—	—	—	—	—	—	1	1	2	2	8	2	2	—	—	—	—	—	18
Totals	3	10	23	21	51	87	104	120	203	166	164	148	178	87	67	45	23	20	3	9	1532

TABLE SS. *Cross-Homotyposis of Mottling and Ground-Colour.*

Mottling of First Egg.

Ground-Colour of Second Egg.	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*	Totals
2	4	1	—	1	7·25	1	1	—	2·75	—	—	—	18
3	15	9	13·5	1	8	5	2	2	·5	2	—	—	58
4	10	10	8·5	—	15·5	6	1	2	2	4	—	—	59
A_1	—	—	—	—	—	—	—	—	—	—	—	—	0
A_2	4	5	4	2	8	11	9	5	1	1	1	—	51
A_3	45·5	19	11·5	4·5	27·5	22	16·5	6	5	1	1	—	159·5
A_4	9	6	4·5	·5	6·5	3	·5	3	—	2	—	1	36
B_1	3	4	4	2	—	3	—	—	—	—	—	—	16
B_2	21	7	3	2	6	10·5	5·5	2	5	—	—	—	62
B_3	49·5	27·5	33·5	5·5	34·25	35·5	9·5	4	12·25	3·5	2	2	219
B_4	14·5	4	9·5	2	9	2	1·5	1	—	1	—	2	46·5
C_1	·5	·5	2·5	—	1·5	—	1	2	—	—	—	—	8
C_2	43	14	13·5	7·5	14	17·5	10	5·5	5	2	1	—	133
C_3	59·25	21	31	8·5	35	20·5	17·5	4·5	7·75	3	1	—	209
C_4	22·75	6·5	1·5	2	11·75	4·5	3	1	4·5	3·5	—	—	61
D_1	—	—	—	—	—	2	—	—	—	—	—	—	2
D_2	13·25	15	3	3	14·75	11	2	1	1	2	1	—	67
D_3	45·75	34·5	34·5	8·5	35·5	28	10	9	8·75	7·5	5	4	231
D_4	22·5	10	6	6	18·5	9·5	1·5	3	—	4	—	1	82
*	3·5	·5	4	—	2	—	3	—	1	—	—	—	14
Totals	386	194·5	188	56	255	192	94·5	51	56·5	36·5	12	10	1532

* Character unrecorded.

ON THE DEGREE OF ASSOCIATION BETWEEN REACTION TIMES IN THE CASE OF DIFFERENT SENSES.

By Professor Y. KOGA,

WITH THE ASSISTANCE OF

G. M. MORANT, M.Sc.*

(1) *Introductory.* It seems of interest to investigate on adequate material the relative extent to which a short reaction time depends on the acuteness of the sense concerned, and on what we may term mental briskness. If reaction times to sight and sound depend largely on acuteness of sight and acuteness of hearing the correlations between acuteness and reaction time will be high; if, on the other hand, mentality is the chief factor in reaction time, we should expect a high correlation between reaction times for the different senses. Such is the problem which the present paper proposes to discuss.

There are, however, considerable difficulties in the path. In the first place, there is the need for adequate material, and the determination of reaction times is rather a slow process in an anthropometrical laboratory. Secondly, not only the reaction times but the acutenesses of the senses are by no means straightforward functions of age, and correction cannot be made by any such simple process as that of partial correlation. As a matter of fact all the regression curves are non-linear, and at first sight by no means simple in character. To have collected *de novo* the adequate material—several thousand cases—would have required much labour, especially if it had been thought desirable to limit the observations to persons approximately of the same age.

Luckily the Galton Laboratory is in possession of the measurements taken in Galton's Anthropometric Laboratory over the course of a number of years. They involve observations on over 9000 persons of both sexes, and ranging in age from quite young children to the aged. These observations have never been reduced as a whole, but have recently been transferred from the record books to cards preparatory to reduction by members of the laboratory staff. The cards were placed at Professor Koga's disposal by the Director of the Laboratory, who suggested the

[* Owing to the laborious tabulating and reduction work involved in this paper, Professor Koga was unable to complete it before returning to Japan, and the task of writing up his work fell upon the Editor, who in the course of his examination found that by an oversight the reduction to age 40·5 years had been made by subtracting the corresponding quantities before reducing them from working to actual units. The latter portion of the investigation—the relation of the characters studied after correction for age—had to be re-done. Luckily Mr G. M. Morant consented to undertake this toilsome revision and the material involved in Tables XIII, XV, XVII, XIX, XXI and Diagrams V to XII is due to his aid. While the conclusions drawn from these reductions do not differ widely from those which flowed from Professor Koga's own reductions, it is only right to say in Professor Koga's absence that the responsibility for them must fall on other shoulders. Miss Ida Maclearn has exhibited her usual felicity in the preparation of the diagrams. Ed.]

investigation of the above problem. During the years in which Galton's Anthropometric Laboratory was open, considerable changes were made in the methods of recording both the acuteness of the senses, and the length of the reaction times. It was therefore desirable to use only records obtained in the same uniform manner. As the work of forming the tables was laborious and Professor Koga had only a limited time available in England he confined his attention to the male data. It is to be hoped that the female data may be reduced later in a similar manner for confirmatory purposes and also to determine sexual differences. After the limitations referred to above the investigator found himself in possession of the record cards of 3379 males, which form the material from which conclusions will be drawn. While a large number of physical and sensory characters were measured in Galton's first and second laboratories this paper confines itself to the following as sufficing for the immediate purpose:

(*a*) Age, as deduced from date of birth and date of measurement.

(*b*) Acuteness of hearing in thousands of vibrations per second heard as a note; determined by Galton's whistle.

(*c*) Acuteness of sight as determined by the number of inches at which the unaided eye could read diamond type*. We did not use the keenness of vision as determined by Snellen's types, nor the keenness of hearing records which were also provided.

(*d*) Reaction time to sound.

(*e*) Reaction time to sight.

Both (*d*) and (*e*) were ascertained by Galton's simple pendulum chronograph. This consists of a fairly massive seconds pendulum, which can be released at an angle of 18° from the vertical; during its descent the pendulum gives a sight-signal by brushing against a very light and small mirror which reflects a light off or onto a screen, or, on the other hand, it gives a sound-signal by a light weight being thrown off the pendulum by impact with a hollow box. The position of the pendulum at either of these occurrences is known. The position of the pendulum when the response is made is obtained by means of a thread stretched parallel to the axis of the pendulum by two elastic bands one above and one below, the thread being in a plane through the axis of the pendulum, perpendicular to the plane of the pendulum's motion. This thread moves freely between two parallel bars in a horizontal plane, and the pressing of a key causes the horizontal bars to clamp the thread. Thus the clamped thread gives the position of the pendulum on striking the key. The elastic bands provide for the pendulum not being suddenly checked on the clamping. The horizontal bars are just below a horizontal scale, 800 mm. below the point of suspension of the pendulum. Galton provided a table for reading off the distance along the scale from the vertical position of the pendulum in terms of the time taken from the vertical position to the position in which the thread is clamped†.

* The apparatus is figured on Plate XIII, *Journal of Anthropological Institute*, Vol. XIV; see also pp. 10, 11.

† See the *Journal of the Anthropological Institute*, Vol. XXIX. p. 28, 1889, or Pearson, *Life, Letters and Labours of Francis Galton*, Vol. II. p. 219.

It would seem *a priori* that keenness of hearing would have more relation to reaction time to sound than to the highest audible note, but owing to noises in the laboratory Galton considered the keenness of hearing results to be unsatisfactory and accordingly they were not reduced. The highest audible note is here taken as some measure of the fineness of the ear as a sensory organ. The Snellen's types for keenness of vision are also not very satisfactory for statistical purposes as they do not break up the population into an adequate number of small groups. Still we should have analysed these records of keenness of vision had time permitted.

We propose first to consider the relation of age to all our characters.

(2) *Age and Highest Audible Pitch.* Table I gives our data. The ages are taken in three year periods, the audible pitches in units of 2000 vibrations per second.

Age (x)		Highest audible pitch (y)	
Mean ...	... 24·3119 years.	Mean ...	... 19,080·94 vibrations per sec.
Standard deviation 10·9606	,,	Standard deviation 5009·81	,, ,,

TABLE I. *Correlation of Age and Highest Audible Pitch.*

Age in years

Highest Audible Pitch in thousands of vibrations per second	5·5—8·5	8·5—11·5	11·5—14·5	14·5—17·5	17·5—20·5	20·5—23·5	23·5—26·5	26·5—29·5	29·5—32·5	32·5—35·5	35·5—38·5	38·5—41·5	41·5—44·5	44·5—47·5	47·5—50·5	50·5—53·5	53·5—56·5	56·5—59·5	59·5—62·5	62·5—65·5	65·5—68·5	68·5—71·5	71·5—74·5	74·5—77·5	77·5—80·5	Totals
5—6*	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	3
7—8	—	—	1	1	3	2	—	3	1	1	—	2	1	3	1	3	3	5	2	4	3	2	1	3	—	45
9—10	—	—	—	—	—	—	1	—	—	—	1	—	—	2	2	—	1	1	—	—	1	—	—	—	1	10
11—12	—	—	1	2	2	3	2	4	2	4	2	5	7	6	12	14	5	11	6	8	3	2	—	2	1	104
13—14	—	—	2	3	7	2	1	4	7	5	10	8	4	6	11	4	6	2	4	4	2	—	1	—	—	93
15—16	—	2	7	21	40	32	33	27	18	20	21	22	19	15	16	6	5	3	1	1	1	—	—	—	—	310
17—18	3	4	26	49	121	105	72	50	36	28	21	21	18	13	4	3	2	—	—	—	—	—	—	—	—	576
19—20	3	23	74	153	261	178	110	103	47	34	25	18	13	3	3	1	—	1	1	—	—	—	—	—	—	
21—22	9	39	112	158	303	164	68	47	24	21	6	2	3	1	—	—	—	—	—	—	—	—	—	—	—	
23—24	—	10	27	37	50	22	11	4	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	165
25—26	—	4	5	14	11	3	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	41
27—28	—	1	4	3	6	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
29—30	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
31—32	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
33—34	—	1	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
Totals	15	84	262	442	805	514	301	244	138	115	87	78	65	49	49	31	22	24	14	17	10	4	2	5	2	3379

* Here as elsewhere 5—6 stands for 4·995—6·995 or a central value 5·995.

The correlation between highest audible pitch and age was

$$-\cdot6136 \pm \cdot0072,$$

a very substantial correlation indicating that the power to appreciate high pitches decreases fairly rapidly with age. It was noted at once, however, that the regression is very sensibly skew. We have:

$$\eta_{x \cdot y}, \text{ i.e. age on pitch} = \cdot6522 \pm \cdot0067,$$
$$\eta_{y \cdot x}, \text{ i.e. pitch on age} = \cdot6231 \pm \cdot0071.$$

It is the latter regression which is for our purposes important, and it was accordingly fitted with a cubic by least squares. Taking the origin at age 22 and pitch 19,500,

the units being 3 years for age and 2000 vibrations for pitch, the equation to this cubic is:

$$y = \cdot 085{,}365 - \cdot 204{,}318x - \cdot 013{,}269x^2 + \cdot 000{,}421x^3.$$

This cubic is plotted to the data in Diagram I*. It will be seen that the fit is quite good; the deviations in old age are not more than might be anticipated from the sparseness of the data and the probability that some of the very deaf aged refrained from subjecting themselves to the test.

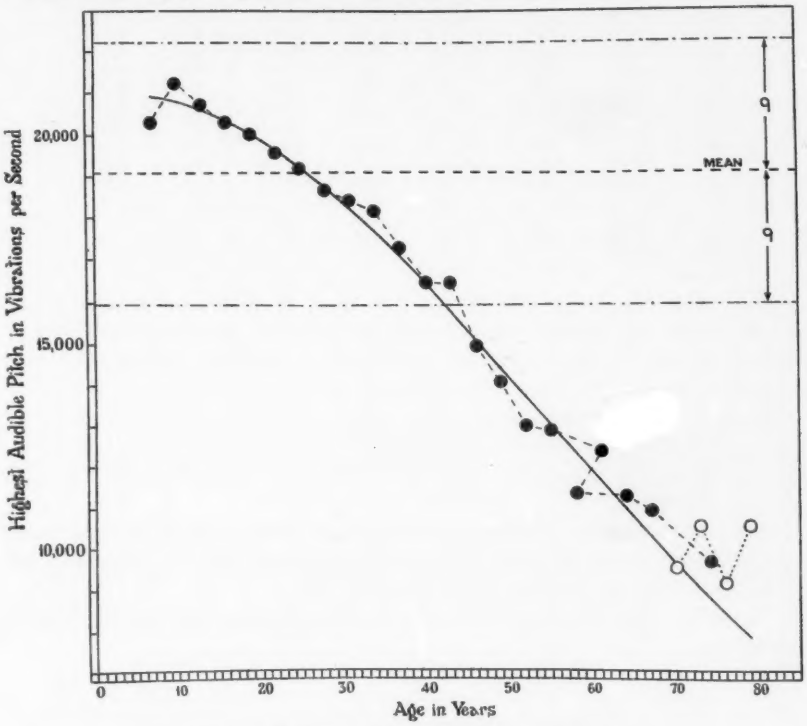


Diagram I. Change of Acuity of Hearing with Age.

We now took an adult male in his forty-first year, i.e. at central age 40·5, as our standard, and by means of the curve for average change of pitch with age reduced all our individuals to their probable highest audible pitch at 40·5 years. The following table (Table II) gives the probable highest audible pitch at each age.

* In all the diagrams of this paper the reader will note two matters: (i) points deduced from small frequencies near the terminals of the range are marked by *clear* circles, and a heavy circle is shown replacing several such points, (ii) the standard deviation σ of the dependent variate is always indicated so that the reader can form a just if rapid appreciation of the extent of the total variation of one character with the other.

TABLE II. *Probable Highest Audible Pitch at each Age.*

Age	Pitch	Age	Pitch	Age	Pitch	Age	Pitch
5·5	20,975	25·5	19,159	45·5	15,246	65·5	10,734
6·5	20,957	26·5	19,001	46·5	15,022	66·5	10,519
7·5	20,931	27·5	18,838	47·5	14,797	67·5	10,307
8·5	20,895	28·5	18,669	48·5	14,571	68·5	10,098
9·5	20,852	29·5	18,496	49·5	14,344	69·5	9,891
10·5	20,800	30·5	18,319	50·5	14,116	70·5	9,688
11·5	20,740	31·5	18,137	51·5	13,887	71·5	9,487
12·5	20,672	32·5	17,952	52·5	13,658	72·5	9,290
13·5	20,596	33·5	17,762	53·5	13,429	73·5	9,096
14·5	20,513	34·5	17,568	54·5	13,200	74·5	8,907
15·5	20,423	35·5	17,371	55·5	12,971	75·5	8,721
16·5	20,325	36·5	17,171	56·5	12,743	76·5	8,539
17·5	20,221	37·5	16,967	57·5	12,515	77·5	8,362
18·5	20,111	38·5	16,761	58·5	12,288	78·5	8,189
19·5	19,992	39·5	16,551	59·5	12,061	79·5	8,026
20·5	19,868	40·5	16,339	60·5	11,836	80·5	7,857
21·5	19,738	41·5	16,125	61·5	11,612	81·5	7,699
22·5	19,602	42·5	15,908	62·5	11,390		
23·5	19,460	43·5	15,689	63·5	11,169		
24·5	19,312	44·5	15,469	64·5	10,951		

This table is extremely instructive as illustrating the rapidity with which acuity of hearing fails with increasing age. The English proverb, "No Dorsetshire labourer, when he is over 40 years old, can hear a bat's cry," suggests that the pitch of a bat's cry is about 16,000 vibrations per second as estimated on the Galton whistle.

On the record cards were recorded the highest audible pitches in 1000 vibrations per second, and by aid of the above table, by addition or subtraction of the differences, we reduced the recorded pitch of the individual to the probable highest audible pitch at 40·5 years.

This will be spoken of as pitch corrected for age. Such a procedure seems more adequate than assuming linearity of regression and proceeding by partial linear correlations, when the regression is so obviously skew.

Table III shows the calculated and observed highest audible pitches at each central age.

TABLE III. *Pitches at Central Ages.*

Central age	Observed	Computed	Central age	Observed	Computed
7	20,300	20,945	46	14,929	15,134
10	21,262	20,827	49	14,071	14,457
13	20,721	20,635	52	12,984	13,773
16	20,301	20,375	55	12,864	13,085
19	20,044	20,052	58	11,333	12,403
22	19,578	19,671	61	12,352	11,723
25	19,161	19,236	64	11,265	11,059
28	18,689	18,754	67	10,900	10,412
31	18,413	18,229	70	9,500	9,788
34	18,126	17,665	73	10,000	9,191
37	17,247	17,069	76	9,100	8,628
40	16,449	16,475	79	10,500	8,102
43	16,454	15,799	82	—	7,619

(3) *Age and Acuity of Vision.* Table IV gives the data, the ages being again taken in three year sub-ranges and the acuity of vision being given by the distance in inches at which the unaided eye could read diamond type; the sub-ranges being three inches.

Age (x)		Acuity of vision (y)	
Mean	24·3119 years.	Mean	18·2675 inches.
Standard deviation	10·9606 „	Standard deviation	6·6690 „

TABLE IV. *Age and Acuity of Vision.*

Acuity of Vision in inches \ Age in years	5·5—8·5	8·5—11·5	11·5—14·5	14·5—17·5	17·5—20·5	20·5—23·5	23·5—26·5	26·5—29·5	29·5—32·5	32·5—35·5	35·5—38·5	38·5—41·5	41·5—44·5	44·5—47·5	47·5—50·5	50·5—53·5	53·5—56·5	56·5—59·5	59·5—62·5	62·5—65·5	65·5—68·5	68·5—71·5	71·5—74·5	74·5—77·5	77·5—80·5	Totals
*3—5**	2	1	7	6	18	16	12	7	10	5	2	7	10	12	23	20	17	16	8	15	9	4	2	4	1	234
6—8	—	5	6	22	28	22	16	10	3	4	7	9	5	1	1	1	—	2	2	—	—	—	—	1	1	146
9—11	1	3	9	15	39	26	13	14	12	6	7	5	3	2	2	2	1	—	1	2	—	—	—	—	—	163
12—14	3	14	17	42	68	46	33	33	18	15	11	5	8	7	2	—	1	3	1	—	—	—	—	—	—	327
15—17	5	17	41	60	83	59	43	33	17	17	13	15	8	5	3	2	1	1	1	—	1	—	—	—	—	425
18—20	4	24	83	107	172	105	71	50	28	28	12	17	11	9	5	3	2	1	1	—	—	—	—	—	—	733
21—23	—	14	61	86	152	97	44	43	13	12	15	11	13	6	6	1	—	—	—	—	—	—	—	—	—	574
24—26	—	4	28	68	153	93	41	35	28	16	9	5	5	6	4	2	—	—	—	—	—	—	—	—	—	497
27—29	—	2	5	27	72	38	24	13	8	10	10	3	1	1	3	—	—	—	—	—	—	—	—	—	—	217
30—32	—	—	4	8	14	11	4	5	1	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	52
33—35	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
36—38	—	—	—	1	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
39—41	—	—	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	4
Totals	15	84	262	442	805	514	301	244	138	115	87	78	65	49	49	31	22	24	14	17	10	4	2	5	2	3379

* Central value of 3—5 is 4 inches exactly.

The correlation is $-\cdot3112 \pm \cdot0105$.

Diagram II, p. 352, however, indicates that the regression is far from linear. We have indeed:

$\eta_{y \cdot x}$, i.e. acuity of vision on age $= \cdot3996 \pm \cdot0096$,

$\eta_{x \cdot y}$, i.e. age on acuity of vision $= \cdot4799 \pm \cdot0089$.

The regression curve of acuity of vision on age was accordingly fitted with a cubic by least squares. If y be acuity of vision measured in three inch units from 16 inches as origin and x be age measured in three year units from age 22, then we find:

$$y = 1\cdot174{,}108 - \cdot025{,}347x - \cdot038{,}327x^2 + \cdot001{,}208x^3,$$

for the equation to a regression cubic.

The smooth is a fairly reasonable one, though again we note that the few observations in extreme age appear to bend upwards *, probably because those unable to see the diamond type at six inches abstained from the test, and accordingly

* The cubic also tends to turn up at about 86·5 years. The explanation given may not be the whole story. Sight depends much on general health, and as only the most vigorous live to be over 80, it is quite possible that the average sight of men above 80 might be better than, or at least not worse than, that of those in the late 70's owing to the process of selection.

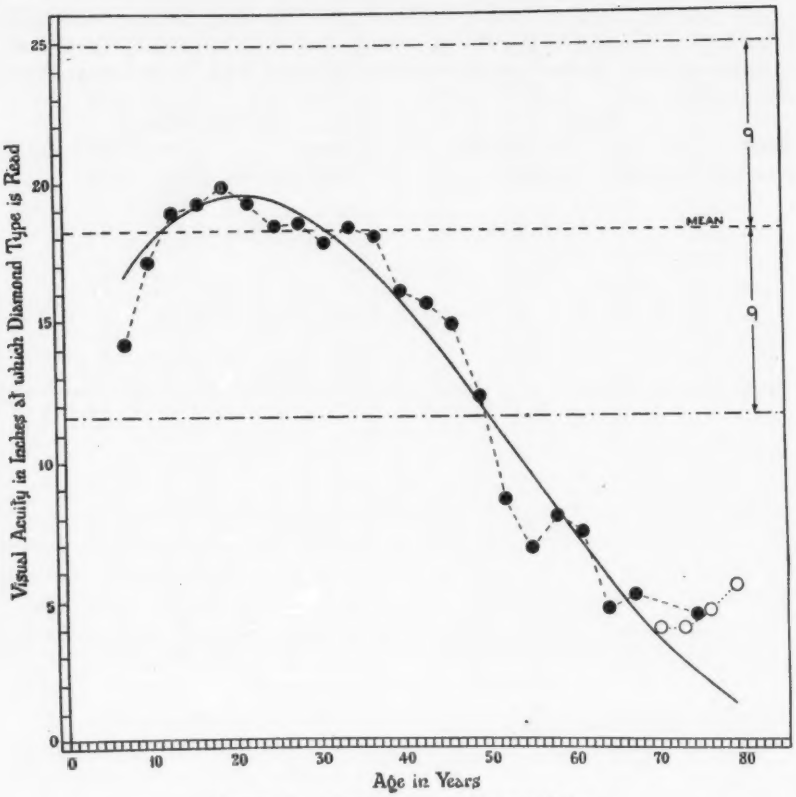


Diagram II. Change of Acuity of Vision with Age.

TABLE V. *Probable Acuity of Vision at each Age.*

Age	Acuity	Age	Acuity	Age	Acuity	Age	Acuity
5·5	15·86	25·5	19·28	45·5	13·61	65·5	5·29
6·5	16·35	26·5	19·16	46·5	13·21	66·5	4·92
7·5	16·79	27·5	19·02	47·5	12·79	67·5	4·56
8·5	17·21	28·5	18·85	48·5	12·38	68·5	4·22
9·5	17·58	29·5	18·67	49·5	11·96	69·5	3·88
10·5	17·92	30·5	18·47	50·5	11·53	70·5	3·56
11·5	18·22	31·5	18·24	51·5	11·10	71·5	3·25
12·5	18·50	32·5	18·00	52·5	10·68	72·5	2·95
13·5	18·73	33·5	17·74	53·5	10·24	73·5	2·67
14·5	18·94	34·5	17·47	54·5	9·81	74·5	2·40
15·5	19·11	35·5	17·18	55·5	9·38	75·5	2·15
16·5	19·25	36·5	16·88	56·5	8·96	76·5	1·92
17·5	19·37	37·5	16·56	57·5	8·53	77·5	1·71
18·5	19·45	38·5	16·23	58·5	8·11	78·5	1·52
19·5	19·50	39·5	15·89	59·5	7·69	79·5	1·34
20·5	19·53	40·5	15·53	60·5	7·27	80·5	1·19
21·5	19·53	41·5	15·17	61·5	6·86	81·5	1·06
22·5	19·51	42·5	14·79	62·5	6·46		
23·5	19·46	43·5	14·41	63·5	6·06		
24·5	19·38	44·5	14·00	64·5	5·67		

were not marked as *nil*. We have included in the table all those who could not read the nearest specimen of diamond type, i.e. that at six inches; they have been put into the group 3 to 5, and this group is accordingly lumped up.

All individuals were reduced to their probable acuity of sight at age 40·5 as standard by aid of Table V calculated from the above cubic. The reductions were entered on the individual cards and the visual acuities so altered will be spoken of as sight corrected for age.

Table V is, we think, of considerable interest. The last column can, of course, have little experimental confirmation because it largely results from the aged who could not see diamond type at any distance at all, or at least not at the lowest distance on the instrument. But as the cubic gives the general sweep of the observations, we think we may conclude that the measures give some rough appreciation of the fall in the acuity of vision even after 65 years of age.

But the noteworthy point here lies in the fact brought out by the data that the maximum acuity of vision is not reached till almost exactly the age of 21, when its average value is 19·56 inches by this test *. In this point sight appears to differ sensibly from acuteness of hearing. In the latter case the youngest ages show the highest audible pitch, although the curve of pitch with age shows some tendency to flatten itself out. In the case of sight the youthful eye is not so keenly acute as that of the young adult, although far more acute than that of the aged. The fact that hearing in man reaches its maximum acuity before sight, suggests that hearing may have had in the past greater evolutionary importance for the young ancestors of man, and it would further suggest not only dealing with the youngest possible ages (i.e. those less than five years) in the case of sight and sound, but also investigating if possible acuity of smell.

Table VI shows the calculated and observed values of visual acuity at each central age, that is, it gives the data on which Diagram II is based.

TABLE VI. *Visual Acuities at Central Ages.*

Central age	Observed	Calculated	Central age	Observed	Calculated
7	14·20	16·58	46	14·84	13·40
10	17·18	17·76	49	12·33	12·16
13	18·93	18·62	52	8·65	10·87
16	19·23	19·18	55	6·59	9·58
19	19·85	19·48	58	8·00	8·29
22	19·29	19·52	61	7·43	7·03
25	18·42	19·33	64	4·71	5·82
28	18·54	18·88	67	5·20	4·68
31	17·83	18·36	70	4·00	3·64
34	18·37	17·61	73	4·00	2·71
37	18·07	16·72	76	4·60	1·92
40	16·04	15·71	79	5·50	1·43
43	15·63	14·60	82	—	1·00

* Stress must be laid on the fact that we are dealing with *average* values at each age. Isolated individuals were in certain cases able to read up to 41 inches.

The reader will observe how much more intensely age affects hearing than sight, the degree of association being as ·6231 to ·3996 or roughly as 3 to 2. If we judged by the coefficient of correlation hearing is almost twice as much influenced by age as sight is. On the whole we think this result accords fairly well with the general experience that delicacy of hearing decays more rapidly with age than acuity of vision.

(4) *Influence of Age on Reaction Time to Sound.* Table VII gives the classified data, the age being measured in three year groups and the reaction time in two hundredths of a second groups.

TABLE VII. *Age and Reaction Time to Sound.*

Reaction Time to Sight in hundredths of a second	Age in years																								
	5·5—8·5	8·5—11·5	11·5—14·5	14·5—17·5	17·5—20·5	20·5—23·5	23·5—26·5	26·5—29·5	29·5—32·5	32·5—35·5	35·5—38·5	38·5—41·5	41·5—44·5	44·5—47·5	47·5—50·5	50·5—53·5	53·5—56·5	56·5—59·5	59·5—62·5	62·5—65·5	65·5—68·5	68·5—71·5	71·5—74·5	74·5—77·5	77·5—80·5	Totals
*9—10**	—	2	4	10	35	28	12	7	4	1	1	3	1	—	3	3	1	2	—	—	—	—	—	—	—	117
11—12	2	—	13	50	91	60	36	33	17	11	7	12	6	5	5	4	2	2	2	1	1	—	1	—	—	361
13—14	—	4	47	107	186	113	70	67	29	27	15	12	12	9	8	6	5	3	1	3	2	—	—	1	—	727
15—16	1	24	91	134	282	165	101	67	44	37	33	28	24	15	18	10	5	7	5	7	1	3	—	2	—	1104
17—18	2	14	47	70	101	80	40	24	26	18	15	9	12	11	11	4	6	5	4	[illegible]	2	1	—	1	2	506
19—20	1	24	43	47	83	41	26	34	14	15	10	9	5	4	3	2	1	5	[illegible]	[illegible]	2	—	—	—	—	375
21—22	—	4	6	12	13	10	7	4	2	3	3	—	1	1	—	—	1	—	[illegible]	[illegible]	1	—	—	—	—	69
23—24	2	6	1	5	9	12	7	6	1	2	2	4	1	1	—	—	1	—	[illegible]	—	—	—	1	—	—	61
25—26	1	6	6	5	3	5	1	1	1	—	1	—	1	3	1	—	—	—	—	—	—	—	—	—	—	35
27—28	3	—	4	2	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	12
29—30	1	—	—	—	1	—	1	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	5
31—32	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
33—34	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
35—36	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	2
37—38	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
39—40	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
41—42	1	—	—	—	—	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	3
Totals	15	84	262	442	805	514	301	244	138	115	87	78	65	49	49	31	22	24	14	17	10	4	2	5	2	3379

* This group is 8·5 to 10·5 or central value 9·5.

The constants determined from this table are as follows:

Age (x)		Reaction time to sound (y)	
Mean	24·3119 years.	Mean	15·6586 $\frac{1}{100}$th sec.
Standard deviation	10·9606 „	Standard deviation	3·2625 $\frac{1}{100}$th „

The correlation coefficient is $-$ ·0088 $\pm$ ·0116.

There is accordingly no significant correlation judged by this coefficient, and we should be easily led astray did we place our trust in the correlation coefficient without further consideration. We find, however, for the correlation ratios:

$\eta_{y \cdot x}$, i.e. reaction time on age = ·2453 $\pm$ ·0109,

$\eta_{x \cdot y}$, i.e. age on reaction time = ·1037 $\pm$ ·0115,

and the graph, Diagram III, shows that the association is markedly skew and of a peculiar kind. The reaction time is very large in childhood, decreases to a minimum at about 25 years of age and then slowly increases with old age. A quite similar phenomenon will be found in our next section dealing with age and reaction time to sight. It became needful accordingly to choose a suitable curve for graduating such relations. The reaction time of a baby at birth is indefinitely great; it does not respond to light or sound. This stage lasts some time, and even when there is response the mental process which dictates an action is usually a very slow one; this hesitation continues

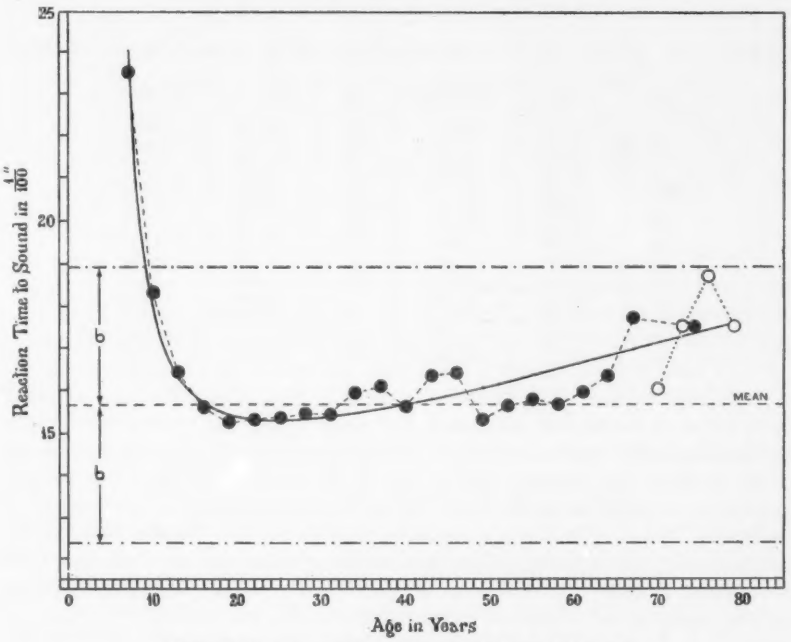


Diagram III. Change of Reaction Time to Sound with Age.

long after infancy *. Without direct experiment it would not be possible to choose the really correct asymptote. But as our sole object was to obtain a reasonable graduation of our data the exact position of the asymptote was not of very great importance. After some trials we found that it was adequate to place the asymptote at four years of age in order to obtain a reasonable smooth. We accordingly adopted a curve of the form:

$$x^2 (y - a - bx) = cx + d,$$

where $x = 0$ is the childhood asymptote and $y = a + bx$ the old age asymptote.

* Delay in executing an order is often treated in a child as disobedience, when it may well be only a very long reaction time to the sound signal.

We took the origin at four years of age ($x = 0$) and at 15·5 units of reaction time and then fitted the cubic

$$y = a + bx + \frac{c}{x} + \frac{d}{x^2}$$

by the method of least squares. The resulting equation was:

$$y = -1{\cdot}29346 + {\cdot}08769x + \frac{3{\cdot}71110}{x} + \frac{1{\cdot}78092}{x^2},$$

where the age unit is three years and the reaction time unit $\frac{3}{100}$ths of a second. From this equation the reaction times at the central ages were calculated and they are given together with the observed values in Table VIII.

TABLE VIII. *Reaction Times to Sound in hundredths of a second for Central Ages.*

Age	Observed	Calculated	Age	Observed	Calculated
7	23·633	24·073	46	16·398	15·917
10	18·310	18·865	49	15·296	16·054
13	16·448	16·309	52	15·629	16·197
16	15·590	15·693	55	15·773	16·343
19	15·244	15·417	58	15·667	16·493
22	15·298	15·300	61	15·929	16·646
25	15·334	15·274	64	16·324	16·813
28	15·434	15·300	67	17·700	16·958
31	15·413	15·360	70	16·000	17·116
34	15·935	15·445	73	17·500	17·276
37	16·075	15·539	76	18·700	17·438
40	15·603	15·661	79	17·500	17·600
43	16·331	15·785	82	—	17·764

It will be seen that the graduation is about as good as is likely to be obtained by a curve of continuous curvature, but there appear to be—considering the numbers dealt with—certain systematic, if irregular deviations. There is a lengthening of the reaction time between ages 34 and 46 and a shortening of it between the ages 49 and 64 which can hardly be due to random sampling considering the numbers dealt with. We are not able to account for these deviations. The deviations in the reaction time to the sight signal hardly fall in line with those for the sound signal and we are too far removed from the sources of the data to endeavour to give any satisfactory explanation *. Short of a sinuous curve with less explicability than our present one, it does not seem possible to obtain a more reasonable smooth. The following table (Table IX) gives the sound reaction times to each age, and was used to reduce all individuals to their sound reaction time at age 40·5 as standard.

This table is of some interest for it shows that while acuteness of hearing is greatest at the earliest ages and while acuteness of sight is greatest at 21 years mental alertness as shown by the minimum reaction time to a sound signal is not reached till the age of 24·75. We shall see that this is very close to the age of minimum reaction time to a sight signal. It would thus appear that the maximum of mental alertness is reached at a later age than the sensory maximum acuities.

* We have to remember that Galton's Laboratory was open for a number of years (1884—1890) and in the earliest years it was in the International Health Exhibition and later in the South Kensington Museum. If the elder men came in the first and the young chiefly in the second period and some slight change had been made in the instrument at the transfer the deviations might be accounted for, but it is not possible now to trace such a change.

TABLE IX. *Reaction Times to Sound in hundredths of a second for each year of Age.*

Age	Reaction time	Age	Reaction time	Age	Reaction time	Age	Reaction time
5·5	42·092	25·5	15·275	45·5	15·894	65·5	16·879
6·5	27·095	26·5	15·281	46·5	15·939	66·5	16·931
7·5	22·096	27·5	15·292	47·5	15·985	67·5	16·984
8·5	19·707	28·5	15·307	48·5	16·031	68·5	17·036
9·5	18·343	29·5	15·326	49·5	16·078	69·5	17·090
10·5	17·473	30·5	15·348	50·5	16·125	70·5	17·143
11·5	16·890	31·5	15·373	51·5	16·173	71·5	17·196
12·5	16·473	32·5	15·400	52·5	16·221	72·5	17·249
13·5	16·167	33·5	15·429	53·5	16·270	73·5	17·303
14·5	15·938	34·5	15·460	54·5	16·319	74·5	17·357
15·5	15·764	35·5	15·494	55·5	16·368	75·5	17·410
16·5	15·630	36·5	15·529	56·5	16·418	76·5	17·465
17·5	15·528	37·5	15·565	57·5	16·468	77·5	17·519
18·5	15·449	38·5	15·601	58·5	16·519	78·5	17·573
19·5	15·389	39·5	15·641	59·5	16·573	79·5	17·627
20·5	15·345	40·5	15·681	60·5	16·620	80·5	17·682
21·5	15·313	41·5	15·722	61·5	16·671	81·5	17·736
22·5	15·292	42·5	15·764	62·5	16·723		
23·5	15·279	43·5	15·807	63·5	16·775		
24·5	15·274	44·5	15·850	64·5	16·827		

A man reaches his sense prime before he reaches his mental prime. From the standpoint of evolutionary history this again seems suggestive, for the senses were certainly developed to a high grade of acuity before the mental powers were. And before a man has reached his highest mental alertness, he has begun to lose the highest grade of sensory acuity, a transition which is probably also characteristic of the evolutionary development of man.

(5) *Influence of Age on Reaction Time to Sight.* Table X gives the data for reaction

TABLE X. *Age and Reaction Time to Sight.*

Age in years

Reaction Time to Sight in hundredths of a second	5·5—8·5	8·5—11·5	11·5—14·5	14·5—17·5	17·5—20·5	20·5—23·5	23·5—26·5	26·5—29·5	29·5—32·5	32·5—35·5	35·5—38·5	38·5—41·5	41·5—44·5	44·5—47·5	47·5—50·5	50·5—53·5	53·5—56·5	56·5—59·5	59·5—62·5	62·5—65·5	65·5—68·5	68·5—71·5	71·5—74·5	74·5—77·5	77·5—80·5	Totals
*9—10**	—	4	6	7	22	26	7	10	4	2	2	2	2	2	1	2	3	1	—	—	—	—	—	—	—	103
11—12	—	2	4	15	38	17	13	8	5	4	2	2	3	1	1	1	—	3	—	1	—	—	—	—	—	120
13—14	1	1	13	28	49	34	14	19	11	9	9	4	2	2	2	—	—	2	—	1	—	—	—	—	—	201
15—16	—	7	33	84	143	87	63	47	36	25	14	17	9	7	10	3	2	2	2	1	2	—	1	—	1	596
17—18	2	10	46	73	156	107	58	49	27	18	18	11	14	9	10	5	8	2	2	3	3	1	—	—	—	632
19—20	2	14	71	114	232	129	84	62	35	32	27	20	17	13	14	7	5	6	4	3	—	2	—	2	1	896
21—22	—	15	34	58	90	48	21	13	8	12	5	6	3	6	7	7	1	3	4	—	1	—	—	—	—	342
23—24	3	13	26	37	37	31	18	17	6	6	2	9	5	3	2	4	2	3	2	4	1	1	—	2	—	234
25—26	1	12	19	15	24	22	16	9	3	5	5	5	6	3	1	1	1	2	—	1	1	—	1	—	—	153
27—28	1	2	4	4	5	5	2	3	1	—	1	2	1	2	—	—	—	—	—	1	1	—	—	—	—	35
29—30	3	2	5	6	7	8	1	5	—	2	1	—	1	—	1	—	—	—	—	2	1	—	—	—	—	45
31—32	—	1	—	1	1	—	3	1	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	1	—	10
33—34	—	1	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
35—36	—	—	1	—	—	—	—	1	1	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	5
37—38	2	—	—	—	1	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	4
Totals	15	84	262	442	805	514	301	244	138	115	87	78	65	49	49	31	22	24	14	17	10	4	2	5	2	3379

* This group is 8·5 to 10·5 or central value 9·5 in hundredths of a second.

times to sight. The ages are given as before in three year groups, and the reaction times in groups of two hundredths of a second. The chief constants are as follows:

Age (x)		Reaction time to sight (y)	
Mean	24·3119 years.	Mean	18·5346 $\frac{1}{100}$ths of a sec.
Standard deviation	10·9606 „	Standard deviation	4·0333 $\frac{1}{100}$ths „

The correlation coefficient is $-\cdot0011 \pm \cdot0116$, and is consequently insignificant. This is again due to the fact that the regression is skew. We have:

$$\eta_{x \cdot y}, \text{ i.e. age on reaction time} = \cdot0651 \pm \cdot0116,$$
$$\eta_{y \cdot x}, \text{ i.e. reaction time on age} = \cdot1936 \pm \cdot0112,$$

and accordingly the regression of reaction time on age is quite significant as is shown in Diagram IV.

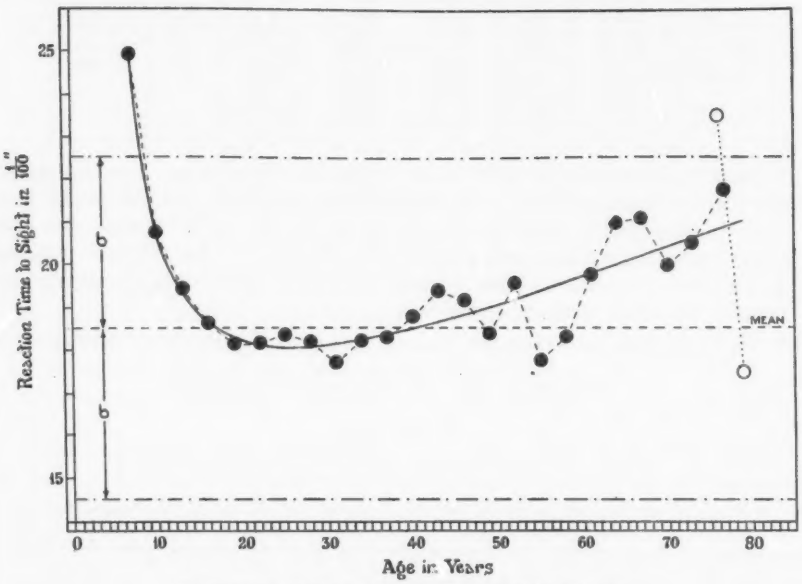


Diagram IV. Change of Reaction Time to Sight with Age.

To smooth these results a curve of the same character as that for reaction time to sound was adopted, taking the childhood asymptote also at four years. The curve fitted by least squares was found to be:

$$y = -2\cdot41234 + \cdot11755x + \frac{6\cdot35462}{x} - \frac{1\cdot32912}{x^2}.$$

The origin is at $y = 19\cdot5$ hundredths of a second and $x = 4$ years, the unit of y being two hundredths of a second and of x three years.

Table XI gives the values deduced from this cubic of the reaction times to sight

of the central ages and the corresponding observed reaction times. There is a certain amount of irregularity at the later ages, but it is largely a result of the relatively small frequencies at the greater ages.

TABLE XI. *Reaction Times to Sight in hundredths of a second for Central Ages.*

Age	Observed	Calculated	Age	Observed	Calculated
7	24·967	24·961	46	19·173	18·861
10	20·762	20·836	49	18·439	19·039
13	19·462	19·322	52	19·565	19·221
16	18·635	18·627	55	17·773	19·410
19	18·146	18·286	58	18·313	19·605
22	18·158	18·130	61	19·786	19·804
25	18·370	18·082	64	21·029	20·006
28	18·213	18·103	67	21·100	20·212
31	17·703	18·171	70	20·000	20·420
34	18·248	18·271	73	21·500	20·630
37	18·305	18·395	76	23·500 } *	20·843
40	18·782	18·537	79	17·500 }	21·057
43	19·408	18·694	82		

* The mean of these seven individuals combined is 21·786 and occurs at 76·86 years. This point is given on Diagram IV.

Table XII gives the probable reaction times to sight at each year of age, and has been used to correct all individuals to their probable reaction times at age 40·5. Of course we are quite conscious that the rate of change in the reaction time with age probably varies from individual to individual but the greatest factor in reaction time is most probably congenital, and we believe that this method of correcting for age is the best that is available under the circumstances.

TABLE XII. *Reaction Times to Sight in hundredths of a second for each year of Age.*

Age	Reaction time	Age	Reaction time	Age	Reaction time	Age	Reaction time
5·5	29·578	25·5	18·082	45·5	18·833	65·5	20·109
6·5	26·295	26·5	18·086	46·5	18·890	66·5	20·177
7·5	23·892	27·5	18·092	47·5	18·948	67·5	20·246
8·5	22·319	28·5	18·112	48·5	19·008	68·5	20·315
9·5	21·248	29·5	18·132	49·5	19·068	69·5	20·385
10·5	20·488	30·5	18·157	50·5	19·128	70·5	20·455
11·5	19·922	31·5	18·185	51·5	19·190	71·5	20·525
12·5	19·496	32·5	18·217	52·5	19·252	72·5	20·595
13·5	19·168	33·5	18·252	53·5	19·315	73·5	20·665
14·5	18·912	34·5	18·290	54·5	19·379	74·5	20·736
15·5	18·711	35·5	18·330	55·5	19·443	75·5	20·807
16·5	18·552	36·5	18·373	56·5	19·507	76·5	20·878
17·5	18·426	37·5	18·417	57·5	19·572	77·5	20·950
18·5	18·327	38·5	18·464	58·5	19·638	78·5	21·021
19·5	18·250	39·5	18·512	59·5	19·704	79·5	21·093
20·5	18·191	40·5	18·562	60·5	19·771	80·5	21·165
21·5	18·148	41·5	18·614	61·5	19·837	81·5	21·237
22·5	18·117	42·5	18·667	62·5	19·905		
23·5	18·096	43·5	18·721	63·5	19·973		
24·5	18·085	44·5	18·776	64·5	20·042		

From the values of the reaction times at 25 (Table XI) and 25·5 (Table XII) we see that the minimum reaction time to sight is slightly under 18·082 hundredths of a second and occurs at 25·25 years of age. It is thus half a year later than the age at which the minimum reaction time to sound occurs, i.e. 24·75. But while the reaction time to sight is shorter in childhood—up to about 7 years—than the reaction time to sound, after childhood the reaction time to sound is always less than the reaction time to sight. While the reaction time to sight is somewhat under 3·00 $\frac{1}{100}$ths of a second greater than the reaction time to sound before 50, after that age the difference becomes greater than 3·00. This remark does not, however, apply to ages between 10 and 14, where the difference exceeds $\frac{3}{100}$ths of a second. These points are rather important, as had the excess of sight reaction time over sound reaction time been approximately the same throughout the range of observation, it might reasonably be attributed to an instrumental retardation in the sight signal. As it is we seem justified in assuming the reaction time to sight to be greater than that to sound for the greater part of life although the difference between the two reaction times varies with the age. The fact that the "old age asymptote" for sight

$$y = -2{\cdot}41234 + {\cdot}11755x$$

is steeper than that for sound

$$y = -1{\cdot}29346 + {\cdot}08769x$$

is in accordance with the more rapid increase of reaction to sight than to sound in the case of the aged.

Tables IX and XII will possibly be of service for comparative purposes when other investigators test groups at a given age for sound or sight reaction times, although something of course depends on similarity of apparatus. Since the mean square root of weighted squared deviations from the regression line is $\sigma_y\sqrt{1-\eta^2_{y.x}}$, we should expect for a group of size n a rough comparative estimate to be obtained by taking out the reaction time R_A for a given age A from our tables and comparing

$$\left.\begin{array}{ll} \text{for sound} & R_A \pm \dfrac{\cdot 67449}{\sqrt{n}} \times 3{\cdot}1628 \\ \text{for sight} & R_A \pm \dfrac{\cdot 67449}{\sqrt{n}} \times 3{\cdot}9570 \end{array}\right\} \text{ in } \tfrac{1}{100}\text{ths of a second}$$

with the comparative results.

While the reaction time to sight is for all but extreme childhood greater than that for sound, the variability as measured by the standard deviations is also greater for sight—approximately in the ratio of 5 to 4. On the other hand, as judged by the coefficients of variation, namely:

sound 20·8352, sight 21·7609,

there is but little difference between them. The coefficients of variation show how extremely variable mental response is, and accordingly what a wide field mentality presents for selection in the case of man, and this is true quite apart from change of reaction time with age.

(6) *On the Association of sensory and mental Characters after Correction for Age.* By means of the age curves in the preceding sections of this memoir, acuity of hearing, keenness of vision, reaction time to sound and reaction time to sight were reduced to the standard age 40·5, and new tables formed for the variables so modified or corrected. It is convenient to have symbols for these corrected variables and we accordingly take:

p = highest audible pitch for individual at age 40·5.
k = keenness of vision " " "
h = reaction time to sound " " "
v = reaction time to sight " " "

We shall investigate the associations of these characters in the following combinations:

(α) Highest audible pitch and keenness of vision = r_{pk}, $\eta_{p \cdot k}$ and $\eta_{k \cdot p}$.

(β) Highest audible pitch and reaction time to sound = r_{ph}, $\eta_{p \cdot h}$ and $\eta_{h \cdot p}$.

(γ) Keenness of vision and reaction time to sight = r_{kv}, $\eta_{k \cdot v}$ and $\eta_{v \cdot k}$.

(δ) Reaction time to sound and reaction time to sight = r_{hv}, $\eta_{h \cdot v}$ and $\eta_{v \cdot h}$.

Before proceeding to this investigation it may be useful to give a table of the means and variabilities of these new variates.

TABLE XIII. *Constants of Variates before and after correction for Age.*

Character		Mean	Standard deviation	Coefficient of variation
Highest audible pitch	All ages	19,081 ± 36·72	3165 ± 25·97	16·587 ± ·139
	At 40·5 (p)	16,384 ± 30·20	2603 ± 21·52	15·887 ± ·133
Keenness of vision	All ages	18·2675 ± ·0774	6·6690 ± ·0547	36·507 ± ·337
	At 40·5 (k)	16·2908 ± ·0747	6·4381 ± ·0532	39·520 ± ·374
Reaction time to sound signal	All ages	15·6586 ± ·0379	3·2625 ± ·0268	20·835 ± ·178
	At 40·5 (h)	15·8032 ± ·0365	3·1465 ± ·0260	19·991 ± ·171
Reaction time to sight signal	All ages	18·5346 ± ·0468	4·0333 ± ·0331	21·761 ± ·186
	At 40·5 (v)	18·7082 ± ·0463	3·9860 ± ·0329	21·306 ± ·184

With regard to this table we may make the following remarks:

(i) While there is a wide difference in variation between keenness of hearing and keenness of vision, the coefficient of variation being double for the latter what it is for the former, there is but slight difference in the variation of the two reaction times. This in itself suggests that the reaction times depend very little on the sensory acuteness.

(ii) The mean of both sensory acutenesses is lowered quite sensibly by reduction to age 40·5 years, but there is no change of any marked character in the reaction times; it is therefore again unlikely that such times are largely dependent on sensory acuteness.

(iii) The absolute variations (standard deviations) of the reaction times are not markedly modified by correction for age. It would thus seem that variability in reaction time depends in the main neither on sensitivity nor age, but on some other factors, probably mental.

(*α*) *Highest Audible Pitch and Keenness of Vision.* Tables XIV and XV give our data before and after correction for age respectively. They yield the following associational constants:

	r_{pk}	$\eta_{p.k}$	$\eta_{k.p}$
All ages	$+\cdot2646\pm\cdot0108$	$\cdot3491\pm\cdot0102$	$\cdot2955\pm\cdot0106$
At age 40·5	$+\cdot0570\pm\cdot0116$	$\cdot0869\pm\cdot0115$	$\cdot0753\pm\cdot0115$

We see again the skewness of the regressions and that the association of highest audible pitch with keenness of vision is greater than that of keenness of vision with highest audible pitch. Both regressions have been fitted with cubics, which are

$$\text{Probable } p = 16\cdot351 + \cdot038{,}831\tilde{k} + \cdot000{,}609\tilde{k}^2 - \cdot000{,}123\tilde{k}^3,$$
$$\text{Probable } k = 16\cdot258 + \cdot135{,}980\tilde{p} - \cdot000{,}910\tilde{p}^2 - \cdot000{,}276\tilde{p}^3.$$

Here $\tilde{k} = k - 16\cdot495$, $\tilde{p} = p - 15\cdot995$, but the probable p and k are measured from the origin of absolute pitch and vision. The unit of pitch is 1000 vibrations per second and of keenness of vision the inches at which diamond type could be read. The two curves are plotted in Diagrams V and VI.

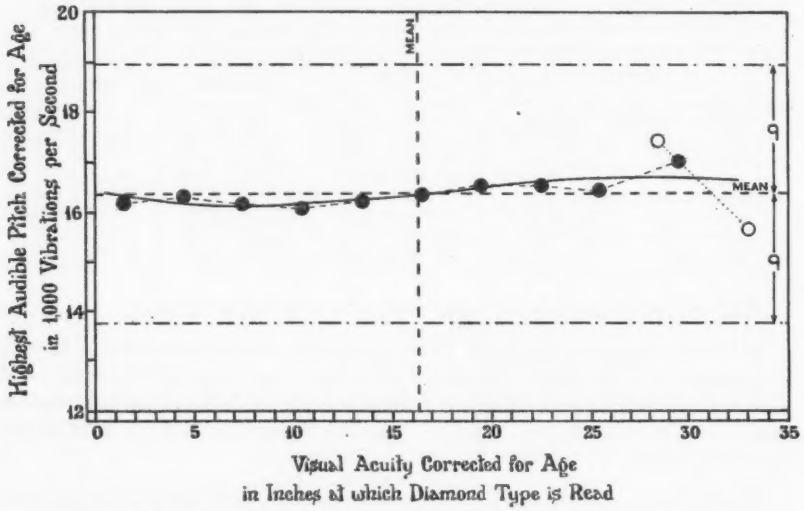


Diagram V. Regression of Acuity of Hearing on Visual Acuity after reduction to Age 40·5 years.

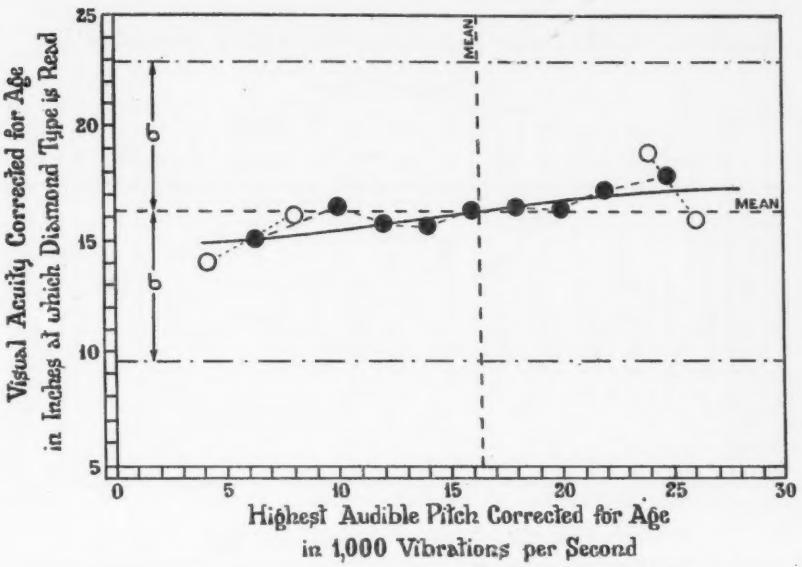


Diagram VI. Regression of Visual Acuity on Acuity of Hearing after reduction to Age 40·5 years.

TABLE XIV. *Highest Audible Pitch and Keenness of Vision, uncorrected for Age.*

Highest Audible Pitch in 1000's of vibrations per sec.	Keenness of Vision*													
	3—5	*6—8*	*9—11*	*12—14*	*15—17*	*18—20*	*21—23*	*24—26*	*27—29*	*30—32*	*33—35*	*36—38*	*39—41*	Totals
5—6	1	—	—	—	2	—	—	—	—	—	—	—	—	3
7—8	22	4	2	3	3	7	2	2	—	—	—	—	—	45
9—10	3	1	—	1	—	1	2	1	1	—	—	—	—	10
11—12	49	3	8	5	8	12	7	10	1	—	—	—	1	104
13—14	32	3	5	9	8	12	10	10	4	—	—	—	—	93
15—16	36	16	16	36	36	64	47	36	18	3	—	2	—	310
17—18	27	26	33	68	79	138	82	75	39	7	—	1	1	576
19—20	33	45	45	102	151	236	191	168	63	16	—	—	1	1051
21—22	25	42	43	81	115	204	191	158	75	20	—	3	—	957
23—24	5	3	10	19	21	37	31	24	11	3	—	—	1	165
25—26	—	1	1	1	2	15	8	7	3	2	—	1	—	41
27—28	1	1	—	—	—	4	2	5	2	1	—	—	—	16
29—30	—	—	—	1	—	—	—	1	—	—	—	—	—	2
31—32	—	—	—	—	—	2	—	—	—	—	—	—	—	2
33—34	—	1	—	1	—	1	1	—	—	—	—	—	—	4
Totals	234	146	163	327	425	733	574	497	217	52	0	7	4	3379

* In inches at which diamond type can be read.

TABLE XV. *Highest Audible Pitch and Keenness of Vision, corrected for Age.*

Highest Audible Pitch in 1000's of vibrations per sec.	Keenness of Vision*																
	0—2	*3—5*	*6—8*	*9—11*	*12—14*	*15—17*	*18—20*	*21—23*	*24—26*	*27—29*	*30—32*	*33—35*	*36—38*	*39—41*	*42—44*	*45—47*	Totals
1—2	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
3—4	1	—	—	1	2	4	1	2	—	—	—	—	—	—	—	—	11
5—6	1	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	3
7—8	—	1	6	—	1	4	2	4	1	—	—	—	—	—	—	—	19
9—10	2	2	6	4	4	12	5	10	4	—	1	—	—	—	—	—	50
11—12	7	14	11	20	37	41	37	21	18	1	2	—	—	—	—	—	209
13—14	18	27	66	43	69	132	83	76	38	8	2	—	1	—	—	—	563
15—16	33	47	73	86	153	244	203	174	69	17	3	—	1	—	—	1	1104
17—18	40	49	84	48	134	206	184	174	82	21	6	—	—	—	—	—	1028
19—20	7	12	30	20	37	61	57	43	17	12	—	1	—	—	—	—	297
21—22	1	2	4	5	6	19	8	10	6	2	1	—	—	—	—	—	64
23—24	1	1	—	—	—	6	3	6	2	1	—	—	—	—	—	—	20
25—26	—	—	—	1	—	2	1	2	—	—	—	—	—	—	—	—	6
27—28	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
29—30	—	1	—	1	—	1	1	—	—	—	—	—	—	—	—	—	4
Totals	111	156	280	230	444	733	585	522	237	62	15	1	2	0	0	1	3379

* In inches at which diamond type can be read.

(β) *Highest Audible Pitch and Reaction Time to Sound.* Tables XVI and XVII give our data before and after correction for age respectively. They yield the following associational constants:

	$r_{p\lambda}$	$\eta_{p.\lambda}$	$\eta_{\lambda.p}$
All ages	$-\cdot0871\pm\cdot0115$	$\cdot1316\pm\cdot0114$	$\cdot1548\pm\cdot0113$
At age 40·5	$-\cdot0956\pm\cdot0115$	$\cdot1481\pm\cdot0114$	$\cdot1406\pm\cdot0114$

The regressions are again skew; the correction for age has not made very substantial changes; and the association of pitch with reaction time is less than that of reaction time with pitch before reduction to age 40·5, but the order is changed after reduction. Both regression curves have been fitted with cubics, but one of the parabolic type was adequate for p on h, but inadequate for h on p. Accordingly for the latter case an asymptotic type was selected. Both were as usual fitted to the weighted means by the method of least squares. Their equations are:

$$\text{Probable } p = 16\cdot387 - \cdot075{,}486\tilde{h} - \cdot001{,}830\tilde{h}^2 + \cdot000{,}014\tilde{h}^3,$$

$$\text{Probable } h = 13\cdot792 + \cdot022{,}925\tilde{p} + \frac{26\cdot01{,}828}{\tilde{p}} - \frac{2\cdot87{,}944}{\tilde{p}^2}.$$

Here the origin is at 15·975 for $\tilde{h}$, and at $-\cdot005$ for $\tilde{p}$, the unit of pitch being 1000 vibrations per second and of reaction time to sound $\frac{1}{100}$th of a second.

The graphs are given in the accompanying Diagrams VII and VIII.

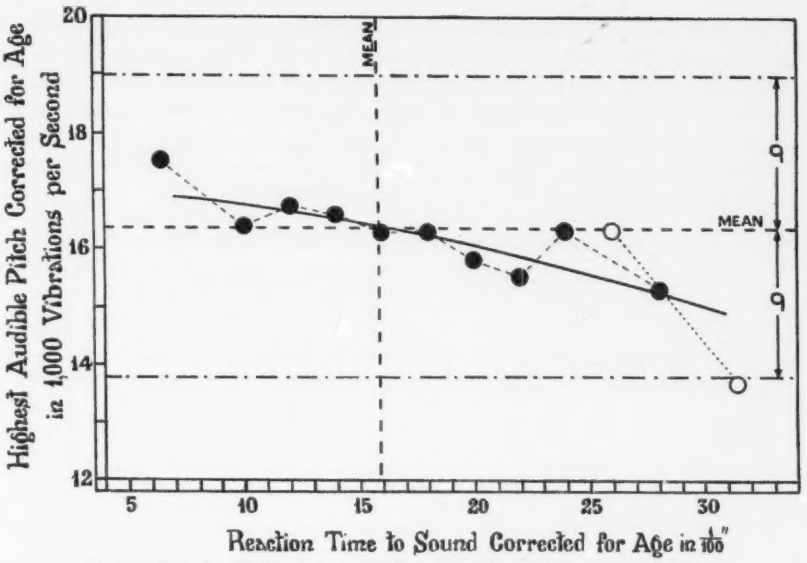


Diagram VII. Regression of Acuity of Hearing on Reaction Time to Sound after reduction to Age 40·5 years.

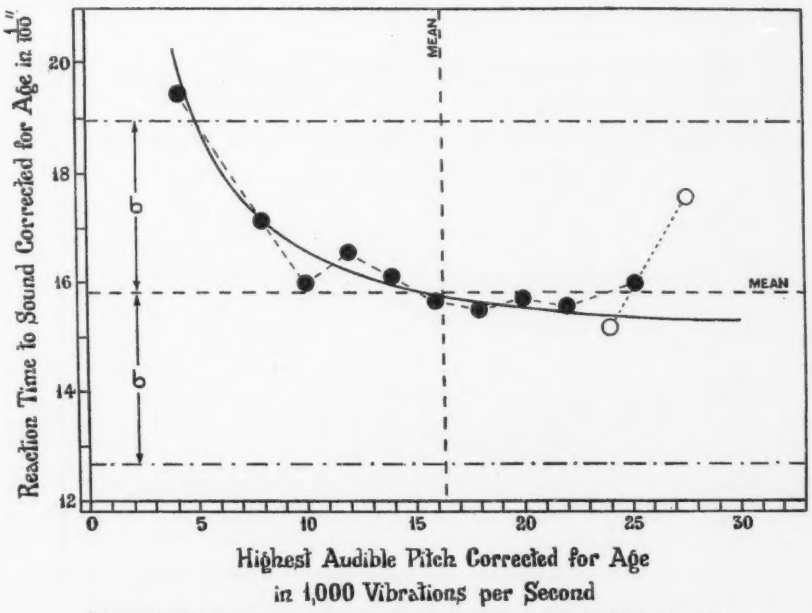


Diagram VIII. Regression of Reaction Time to Sound on Acuity of Hearing after reduction to Age 40·5 years.

We see that corrected for age acuity of hearing becomes slightly less as reaction time increases, the whole range of $\frac{1}{4}$ second in reaction time corresponding to a lowering of highest audible pitch by 2000 vibrations per second. On the other hand as the highest audible pitch increases from 4000 to 10,000 vibrations per second, the reaction time falls very rapidly, and then from 10,000 hardly falls at all; that is to say increased acuity of hearing beyond, say, 15,000 vibrations per second is not accompanied by any real gain in reaction time.

TABLE XVI. *Highest Audible Pitch and Reaction Time to Sound, uncorrected for Age.*

Reaction Time to Sound in $\frac{1}{100}$'s of a second

Highest Audible Pitch in 1000's of vibrations per sec.	*9—10*	*11—12*	*13—14*	*15—16*	*17—18*	*19—20*	*21—22*	*23—24*	*25—26*	*27—28*	*29—30*	*31—32*	*33—34*	*35—36*	*37—38*	*39—40*	*41—42*	Totals
5—6	—	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	3
7—8	—	1	6	16	4	6	4	2	—	2	2	1	—	—	—	—	1	45
9—10	—	1	1	4	3	1	—	—	—	—	—	—	—	—	—	—	—	10
11—12	8	7	13	36	23	14	1	—	1	—	—	—	—	1	—	—	—	104
13—14	3	5	17	21	21	17	3	5	1	—	—	—	—	—	—	—	—	93
15—16	11	32	49	102	53	44	7	6	4	1	—	—	—	—	—	—	1	310
17—18	16	62	112	205	84	65	10	13	5	3	1	—	—	—	—	—	—	576
19—20	37	129	255	327	149	99	25	16	10	2	—	—	—	1	—	—	1	1051
21—22	32	96	224	322	142	101	16	10	10	3	—	—	—	—	—	1	—	957
23—24	8	17	33	55	20	20	3	6	2	—	1	—	—	—	—	—	—	165
25—26	1	9	3	8	5	6	—	2	—	1	1	—	—	—	—	—	—	41
27—28	1	2	6	5	1	—	—	—	1	—	—	—	—	—	—	—	—	16
29—30	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	2
31—32	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	2
33—34	—	—	1	2	—	1	—	—	—	—	—	—	—	—	—	—	—	4
Totals	117	361	727	1104	506	375	69	61	35	12	5	1	0	2	0	1	3	3379

TABLE XVII. *Highest Audible Pitch and Reaction Time to Sound, corrected for Age.*

Reaction Time to Sound in $\frac{1}{100}$'s of a second

Highest Audible Pitch in 1000's of vibrations per sec.	*3—4*	*5—6*	*7—8*	*9—10*	*11—12*	*13—14*	*15—16*	*17—18*	*19—20*	*21—22*	*23—24*	*25—26*	*27—28*	*29—30*	*31—32*	*33—34*	*35—36*	*37—38*	*39—40*	*41—42*	Totals
1—2	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
3—4	—	—	—	—	—	1	2	1	4	1	—	—	1	—	—	—	—	—	—	1	11
5—6	—	—	—	—	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	3
7—8	—	—	—	1	—	5	6	2	3	—	1	—	—	1	—	—	—	—	—	—	19
9—10	—	—	—	3	4	13	13	9	6	1	—	—	1	—	—	—	—	—	—	—	50
11—12	—	—	—	6	22	38	63	29	34	7	6	4	—	—	—	—	—	—	—	—	209
13—14	—	—	—	24	54	125	188	77	57	15	15	3	2	2	—	1	—	—	—	—	563
15—16	2	—	2	47	131	290	342	133	104	28	16	8	—	—	—	—	1	—	—	—	1104
17—18	1	1	2	39	129	261	338	149	77	14	13	4	—	—	—	—	—	—	—	—	1028
19—20	—	—	—	13	32	90	84	41	22	4	9	1	—	—	—	—	—	—	1	—	297
21—22	—	—	1	3	14	16	14	6	5	—	2	—	1	2	—	—	—	—	—	—	64
23—24	—	—	—	1	3	7	6	1	1	—	—	1	—	—	—	—	—	—	—	—	20
25—26	—	—	—	—	1	1	—	1	1	—	1	1	—	—	—	—	—	—	—	—	6
27—28	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
29—30	—	—	—	—	1	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	4
Totals	3	1	5	137	391	849	1059	450	315	70	63	22	5	5	0	1	1	0	1	1	3379

(γ) *Keenness of Sight (or Visual Acuity) and Reaction Time to Sight.* Tables XVIII and XIX give our data before and after correction for age respectively. They yield the following associational constants:

	r_{kv}	$\eta_{k.v}$	$\eta_{v.k}$
All ages	−·1121 ± ·0115	·1404 ± ·0114	·1254 ± ·0114
At age 40·5	−·0644 ± ·0116	·1101 ± ·0115	·1055 ± ·0115

The regression curves are again better treated as skew. The corrections for age are not very considerable, and it is now the sensory acuteness on the reaction time which has the greater correlation ratio *. Again both regression curves have been fitted with cubics, which appear to describe the relationship adequately. They are:

$$\text{Probable } k = 16{\cdot}253 - {\cdot}097{,}667\tilde{v} - {\cdot}007{,}178\tilde{v}^2 - {\cdot}000{,}816\tilde{v}^3,$$

$$\text{Probable } v = 18{\cdot}781 - {\cdot}048{,}214\tilde{k} - {\cdot}002{,}265\tilde{k}^2 - {\cdot}000{,}0865\tilde{k}^3.$$

Here the origin is at 19·995 for $\tilde{v}$ and at 16·495 for $\tilde{k}$, while the unit of keenness of sight is one inch and of reaction time to sight $\frac{1}{100}$th of a second.

The two curves are given in Diagrams IX and X (p. 368). The influence of keenness of sight on reaction time to sight is relatively small after correction for age.

TABLE XVIII. *Visual Acuity and Reaction Time to Sight, uncorrected for Age.*

Visual Acuity *

Reaction Time to Sight in $\frac{1}{100}$'s of a second	3—5	6—8	9—11	12—14	15—17	18—20	21—23	24—26	27—29	30—32	33—35	36—38	39—41	Totals
9—10	5	4	5	5	12	24	17	12	12	4	—	2	1	103
11—12	7	8	3	14	12	32	20	13	9	1	—	—	1	120
13—14	7	8	12	15	34	35	36	33	12	8	—	1	—	201
15—16	34	18	26	61	67	141	89	110	38	11	—	1	—	596
17—18	32	23	43	55	80	133	114	95	49	6	—	1	1	632
19—20	70	39	43	82	122	163	175	140	52	10	—	—	—	896
21—22	27	14	12	33	39	88	52	49	22	4	—	1	1	342
23—24	24	14	9	33	23	50	38	24	15	4	—	—	—	234
25—26	11	9	7	15	21	40	24	15	6	4	—	1	—	153
27—28	7	4	1	5	3	9	3	1	2	—	—	—	—	35
29—30	5	2	2	6	8	14	5	3	—	—	—	—	—	45
31—32	1	1	—	1	1	3	1	2	—	—	—	—	—	10
33—34	—	—	—	—	3	—	—	—	—	—	—	—	—	3
35—36	1	1	—	2	—	1	—	—	—	—	—	—	—	5
37—38	3	1	—	—	—	—	—	—	—	—	—	—	—	4
Totals	234	146	163	327	425	733	574	497	217	52	0	7	4	3379

* In inches at which diamond type can be read.

* In the case of sound it was the reaction time on the sensory acuteness, which had the higher correlation ratio.

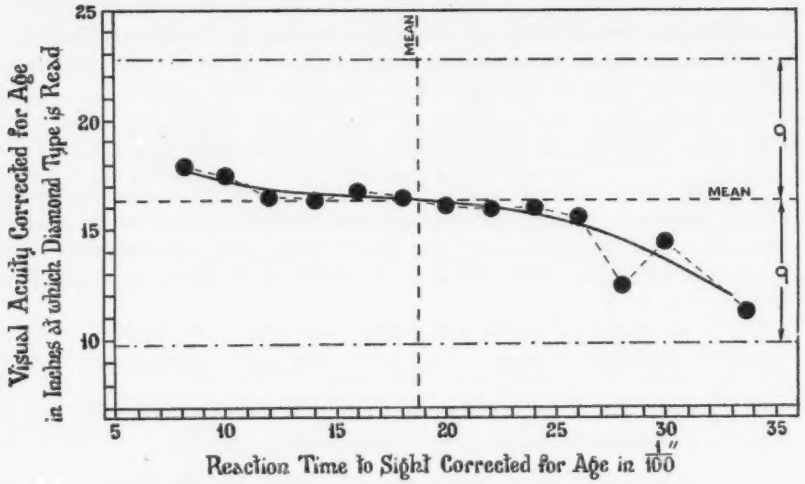


Diagram IX. Regression of Visual Acuity on Reaction Time to Sight after reduction to Age 40·5 years.

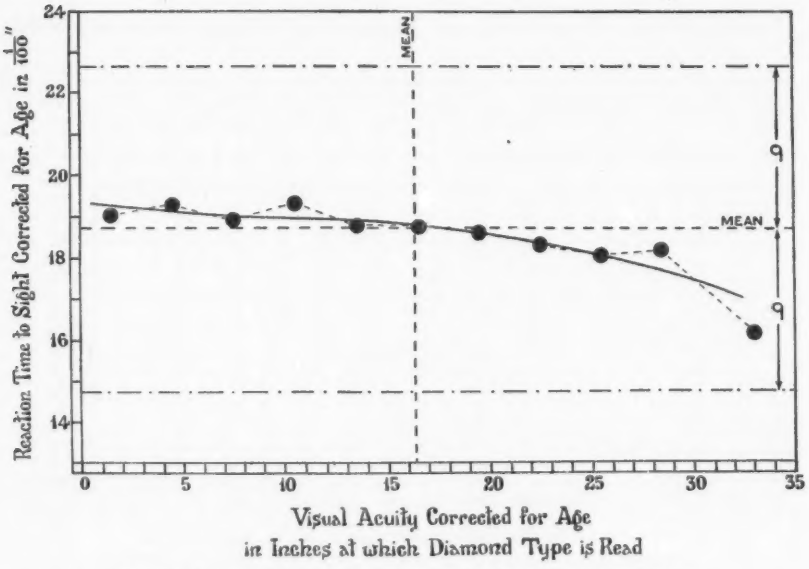


Diagram X. Regression of Reaction Time to Sight on Visual Acuity after reduction to Age 40·5 years.

TABLE XIX. *Visual Acuity and Reaction Time to Sight, corrected for Age.*

Reaction Time to Sight in $\frac{1}{100}$'s of a second	Visual Acuity *															
	0—2	*3—5*	*6—8*	*9—11*	*12—14*	*15—17*	*18—20*	*21—23*	*24—26*	*27—29*	*30—32*	*33—35*	*36—38*	*39—41*	*42—44*	*45—47*	Totals
5—6	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
7—8	—	1	—	1	3	1	1	—	1	—	—	—	—	—	—	1	9
9—10	4	3	6	5	14	23	19	15	12	4	2	—	—	—	—	—	107
11—12	3	8	11	9	11	28	23	19	8	2	2	—	1	—	—	—	125
13—14	8	13	22	9	41	51	35	42	18	7	1	—	—	—	—	—	247
15—16	20	20	43	44	79	153	98	113	47	14	3	—	—	—	—	—	634
17—18	13	30	57	42	79	134	118	102	47	8	3	—	1	—	—	—	634
19—20	35	39	76	49	121	169	177	138	60	10	2	—	—	—	—	—	876
21—22	14	16	25	27	41	69	46	47	25	7	1	1	—	—	—	—	319
23—24	6	7	22	26	21	50	39	29	12	7	—	—	—	—	—	—	219
25—26	2	8	11	9	17	32	17	11	6	3	1	—	—	—	—	—	117
27—28	2	7	1	3	6	8	4	1	1	—	—	—	—	—	—	—	33
29—30	2	1	3	4	8	11	7	3	—	—	—	—	—	—	—	—	39
31—32	2	1	1	—	1	2	1	2	—	—	—	—	—	—	—	—	10
33—34	—	—	2	—	2	—	—	—	—	—	—	—	—	—	—	—	4
35—36	—	—	—	2	—	1	—	—	—	—	—	—	—	—	—	—	3
37—38	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	111	156	280	230	444	733	585	522	237	62	15	1	2	0	0	1	3379

* In inches at which diamond type can be read.

(δ) *Reaction Time to Sound and Reaction Time to Sight.* Tables XX and XXI give our data before and after correction for age respectively. We have the following associational constants:

	r_{hv}	$\eta_{h.v}$	$\eta_{v.h}$
All ages	$+\cdot4520\pm\cdot0092$	$\cdot5332\pm\cdot0082$	$\cdot5008\pm\cdot0087$
At age 40·5	$+\cdot4662\pm\cdot0091$	$\cdot4931\pm\cdot0084$	$\cdot4733\pm\cdot0087$

We see that relatively little change is made in the correlation ratios by correcting for age. $\eta_{v\,.\,h}$ is so close to r_{hv} that it might be adequate to fit the regression of reaction time to sight on reaction time to sound by a straight line. Reaction time to sound on reaction time to sight has been fitted in the usual way (by least squares) with a cubic. The equations to the regression lines treated as cubics are:

$$\text{Probable } v = 18\cdot730 + \cdot621{,}831\tilde{h} + \cdot011{,}352\tilde{h}^2 - \cdot001{,}358\tilde{h}^3,$$

$$\text{Probable } h = 16\cdot044 + \cdot415{,}995\tilde{v} + \cdot015{,}174\tilde{v}^2 - \cdot000{,}560\tilde{v}^3.$$

The origin for $\tilde{h}$ is 15·995 and for $\tilde{v}$ 19·995, the units being as in previous cases $\frac{1}{100}$ths of a second. These regression curves are given in Diagrams XI and XII, pp. 370–1.

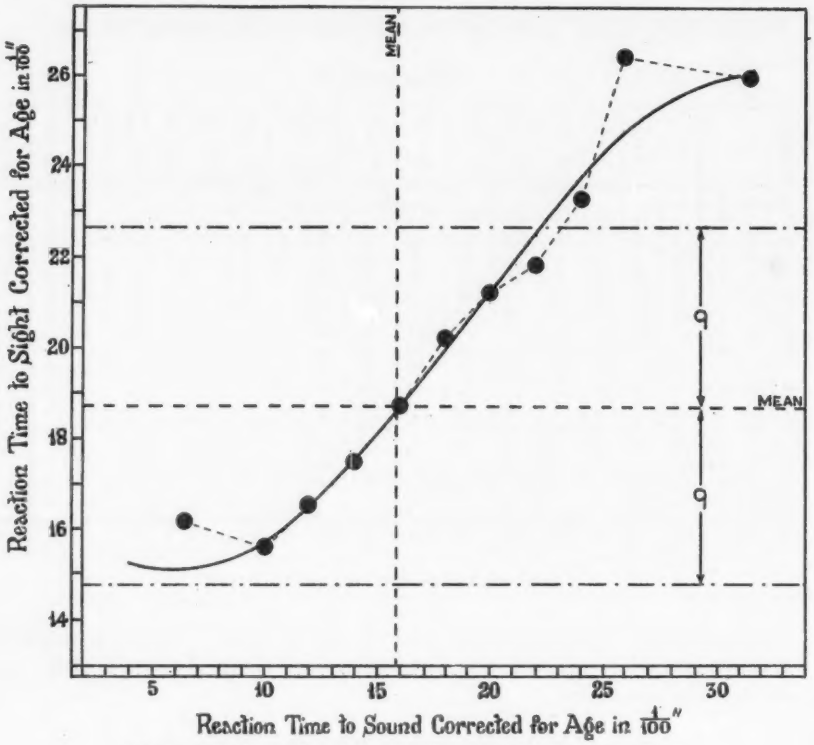


Diagram XI. Regression of Reaction Time to Sight on Reaction Time to Sound after reduction to Age 40·5 years.

TABLE XX. *Reaction Time to Sound and Reaction Time to Sight, uncorrected for Age.*

Reaction Time to Sight in $\frac{1}{100}$'s of a second

Reaction Time to Sound in $\frac{1}{100}$'s of a second	*9—10*	*11—12*	*13—14*	*15—16*	*17—18*	*19—20*	*21—22*	*23—24*	*25—26*	*27—28*	*29—30*	*31—32*	*33—34*	*35—36*	*37—38*	Totals
9—10	14	12	14	29	22	20	2	3	1	—	—	—	—	—	—	117
11—12	16	36	37	111	76	64	9	9	2	—	1	—	—	—	—	361
13—14	25	28	72	179	194	171	28	15	11	1	3	—	—	—	—	727
15—16	34	30	51	182	222	365	137	45	28	5	4	—	—	1	—	1104
17—18	8	11	18	44	74	170	74	58	36	8	3	—	1	1	—	506
19—20	4	3	5	34	30	82	73	70	48	8	15	2	1	—	—	375
21—22	—	—	2	9	9	9	11	16	10	—	2	1	—	—	—	69
23—24	1	—	1	5	3	10	5	10	10	3	8	4	—	—	1	61
25—26	1	—	1	2	—	3	2	7	4	6	5	2	1	1	—	35
27—28	—	—	—	1	—	1	1	1	3	3	2	—	—	—	—	12
29—30	—	—	—	—	1	1	—	—	—	1	—	—	—	1	1	5
31—32	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
33—34	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
35—36	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	2
37—38	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
39—40	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
41—42	—	—	—	—	—	—	—	—	—	—	1	—	—	—	2	3
Totals	103	120	201	596	632	896	342	234	153	35	45	10	3	5	4	3379

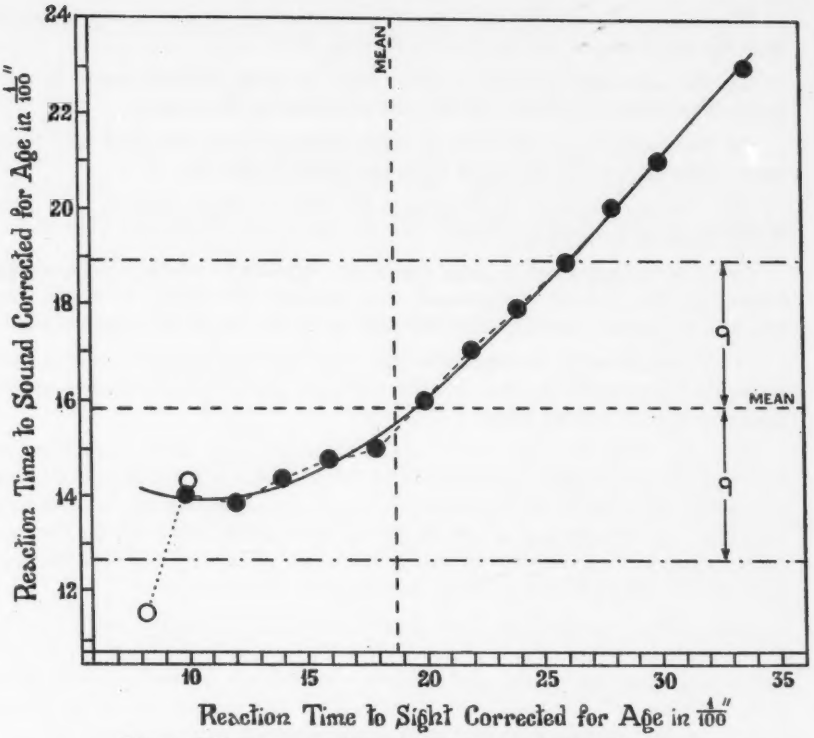


Diagram XII. Regression of Reaction Time to Sound on Reaction Time to Sight after reduction to Age 40·5 years.

TABLE XXI. *Reaction Time to Sound and Reaction Time to Sight, corrected for Age.*

Reaction Time to Sight in $\frac{1}{100}$'s of a second

Reaction Time to Sound in $\frac{1}{100}$'s of a second	5—6	7—8	9—10	11—12	13—14	15—16	17—18	19—20	21—22	23—24	25—26	27—28	29—30	31—32	33—34	35—36	37—38	Totals
3—4	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	3
5—6	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7—8	—	1	—	—	1	—	—	—	2	—	—	—	—	—	—	—	1	5
9—10	1	2	15	13	21	35	26	19	2	2	1	—	—	—	—	—	—	137
11—12	—	1	17	39	46	122	81	64	7	11	1	—	2	—	—	—	—	391
13—14	—	1	30	31	90	207	221	193	44	17	10	2	3	—	—	—	—	849
15—16	—	—	34	28	56	181	208	346	121	53	23	4	3	1	—	1	—	1059
17—18	—	1	7	9	16	43	59	149	65	60	27	10	3	—	—	1	—	450
19—20	—	—	3	3	12	29	27	85	57	47	31	7	11	2	1	—	—	315
21—22	—	—	—	1	2	8	9	8	11	17	7	2	4	1	—	—	—	70
23—24	—	1	1	—	2	6	1	10	9	7	12	2	7	4	—	—	1	63
25—26	—	—	—	—	—	1	1	1	—	5	4	4	3	1	1	1	—	22
27—28	—	—	—	—	—	2	—	—	1	—	—	1	1	—	—	—	—	5
29—30	—	—	—	—	—	—	—	1	—	—	1	1	1	—	1	—	—	5
31—32	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
33—34	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
35—36	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
37—38	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
39—40	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
41—42	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	1	9	107	125	247	634	634	876	319	219	117	33	39	10	4	3	2	3379

We have now to consider what general and important conclusions can be drawn from the above results (α), (β), (γ), (δ). We note that:

(i) The association between reaction times involving different senses is very much closer than the relation between the acuteness of those senses.

(ii) The relation between acuity in different senses is not very high, and more than a third of it is due to change of sensory acuities with age.

(iii) The association of reaction times for different senses depends very little on the common factor of age.

(iv) The reaction times to sense signals are correlated with the corresponding sensory acuities, but the correlation is of a relatively low order. It would thus appear that reaction time depends very little on the fineness of the organs of sense.

(v) We are forced to the conclusion that reaction times are a function of mental rather than sensory effectiveness, and must be taken as a measure rather of mental briskness than of sensory fitness *.

(vi) The whole investigation illustrates the importance of not confining our attention in psychological investigations to the coefficient of correlation and linear regression. Out of the numerous regressions here pictured only one can be effectively described by a straight line; others require at least a cubic, and those for reaction times on age while describable by cubics are not of the usual parabolic form, but need inverse powers as well as direct powers of the primary variate.

(vii) Galton's material seems to provide more ample data for reaction times than have hitherto been discussed, but even with these we have found the records for the higher ages inadequate. But if this be admitted, how hopeless is it to imagine that anything of importance as to reaction time and sensory acuteness can be ascertained from correlations based on 20 to 30 individuals, or even 200 if of different ages! The customary psychological study of small populations—accompanied by very insufficient acquaintance with the theory of small samples—will never succeed in building up any permanent science of psychology.

* We have, it is true, taken measures of the sensory fitness which correspond to sensory factors not necessarily called into play in experiments on the reaction time; they are, however, quite good measures of general sensory fitness. Further this assumes that the time and variation in the central organs are considerably greater than the time and variation in the peripheral organs, which appears in accordance with experience.

STUDIES ON THE SEX-RATIO AND RELATED PHENOMENA.

(4) The Frequencies of Sex Combinations in Pig Litters.

By A. S. PARKES, B.A. Cantab., Ph.D.

CONTENTS

	PAGE
(1) Introductory	373
(2) Size of litter and sex-ratio	374
(3) Sex combinations in litters	377
(4) Summary	381

(1) *Introductory.* In multiple births the sexes may occur in a variety of combinations. Thus in triplets there may be three males, two males and one female, one male and two females, or three females. In larger multiple births the possible combinations are, of course, more numerous. A litter of seven, for instance, may be made up of seven males and no females, six males and one female, five males and two females, etc., down to no males and seven females. If the distribution of the sexes were a matter of chance these combinations would appear in a certain calculable frequency, and the probable frequency may be compared with that actually found. Thus in the human subject, Duncker(1) found that for twins the unisexual births were very much in excess of the probable frequency and that the number of births containing one of each sex was much below the expected. In triplets also the number of unisexual births was found to be almost double the probable, while births containing both sexes had a less frequency than the probable. In sheep, Pearl, as cited, probably erroneously, by Wentworth, found a very irregular and improbable frequency for different combinations, but Wentworth (5) with larger numbers found a frequency very close to the expected. Lillie (2) has considered sex combinations of twins in cattle, but owing to the complication caused by the common occurrence of "free-martins" arrived at no definite results. The cases so far mentioned relate to normally polytocus animals and there is no doubt that polyembryony accounts for the abnormal results observed. In polytocus animals, however, polyembryony can have little influence on the sex constitution of a litter and these cases present a more reasonable field for investigation. Wentworth, working on dogs, found but little divergence from probability in the sex constitution of the litters, and the same was found to hold in the case of pigs. Duncker's work on pigs yielded similar results: "Für die höheren Mehrlinggeburten des Schweins (Drillinge bis Fünflinge) wird die hypothetische Verteilung ihrer Geschlechtskombinationen fast genau identisch mit der wahrscheinlichen" (S. 538), and both the binomial and hypothetical distributions corresponded closely with the observations.

Duncker deals only with litters of three to five, Wentworth's work is also incomplete, and, since it seems that no work existed on the detailed frequency of sex combinations at each litter size the endeavour has been made to provide a more complete investigation. The data can only be obtained from Herd Book records and in most of such cases the total sex constitution of the litter is not given. In the National Duroc-Jersey Pig Record, however, this information is available, and Vol. LXVII of this register was used for the present analysis. Registration statistics of domestic animals are always open to the charge of inaccuracy. This may arise in one or both of two ways. Firstly, the breeder may exterminate the least promising of a litter and register the remainder only, or, secondly, he may take no account of animals which die before registration. Unless, however, the breeder consciously controls the sex constitution of a litter by elimination, and, what is more improbable still, unless certain combinations of the sexes are constantly preserved, it is difficult to see how either of the possible sources of inaccuracy could materially affect the specific point which was the object of this investigation, i.e. the frequency of the different sex combinations at each litter size.

I am under great obligation to Mr Udny Yule, F.R.S., for advice as to method and treatment. To Professor K. Pearson, F.R.S., my thanks are due for allowing me to work in the Biometric Laboratory at University College, while to Mr J. O. Irwin, of that Laboratory, I am much indebted for practical assistance.

(2) *Size of Litter and Sex-ratio.* In order to arrive at the proper basis for calculating the binomial probabilities it is necessary to consider first the gross proportion of males, and also the variation of this proportion in the different sizes of litters. Table I shows the number of litters of each size available. There are 2020 litters in all, varying in size from two to fourteen, the most frequent size being eight. The

TABLE I. *Number of Litters, etc. in Duroc-Jerseys.*

1	2	3	4	5
No. in litter	No. of litters	No. of young	No. of males	Proportion of males
2	9	18	9	·5000
3	24	72	37	·5139
4	53	212	106	·5000
5	116	580	283	·4879
6	221	1326	658	·4962
7	374	2618	1284	·4905
8	402	3216	1581	·4916
9	346	3114	1499	·4814
10	277	2770	1342	·4845
11	102	1122	522	·4652
12	70	840	428	·5095
13	19	247	131	·5304
14	7	98	49	·5000
Totals	2020	16233	7929	·4884
Omitting 2, 3, 13, 14	1961	15798	7703	·4876

average proportion of males is ·4884, which in view of Wodsedalek's account (6) of sex determination in the pig argues some differential mortality during gestation. Owing to the very few litters of 2, 3, 13, and 14, the totals are also given in the same table minus these litters, in which case the mean proportion of males comes out as ·4876.

As the only litters with a preponderance of males are the abnormally sized litters 2, 3, 4, 12, 13, and 14, this table suggests that there is a slight connection between sex-ratio and size of litter, though the small numbers of litters occurring at the extreme sizes prevent too much weight being attached to this.

In an endeavour to push the analysis further the following correlation table was constructed:

TABLE II. *Correlation table for sex-ratio and size of litter.*

		Proportion of Males										Totals	Mean proportion of males
		·0–·1	·11–·2	·21–·3	·31–·4	·41–·5	·51–·6	·61–·7	·71–·8	·81–·9	·91–1·0		
Size of Litter	2	2	—	—	—	5	—	—	—	—	2	9	·5000
	3	4	—	—	7	—	—	9	—	—	4	24	·5139
	4	1	—	14	—	23	—	—	14	—	1	53	·5000
	5	2	20	—	41	—	35	—	14	—	4	116	·4879
	6	3	16	—	53	78	—	53	—	18	—	221	·4962
	7	—	21	63	—	117	104	—	46	21	2	374	·4905
	8	1	8	37	81	162	—	77	30	5	1	402	·4916
	9	—	2	23	72	101	83	46	12	7	—	346	·4814
	10	7	8	19	79	82	48	24	10	—	—	277	·4845
	11	1	3	15	15	33	13	12	8	2	—	102	·4652
	12	—	1	8	4	27	11	15	4	—	—	70	·5095
	13	—	1	—	2	1	9	6	—	—	—	19	·5304
	14	—	—	—	—	6	1	—	—	—	—	7	·5000
Totals		21	80	179	354	635	304	242	138	53	14	2020	·4884
Mean size of litter		6·5328	7·0375	8·1061	8·1045	8·1858	8·3421	8·2851	7·4783	7·1698	4·4286	8·0361	—

A glance at this table disposes of the possibility of linear regression of sex-ratio upon size of litter, but suggests a parabolic correlation. At the same time it is clear, firstly, that the arbitrary grouping of the sex-ratios—·0–·1, ·11–·2, etc.—is unsatisfactory because the most frequent sex-ratios tend to fall away from the centres of such groups. The most frequent sex-ratio of all, for instance, ·5000, falls right at the higher extremity of the group ·41–·5. Secondly, it is equally clear that there is something peculiar about the low mean size of litter in the two extreme sex-ratio groups, and this is due, of course, to the fact that the probability of getting a sex-ratio of between ·0–·1, or between ·91–1·0, is much greater in the case of the smaller litters than in the case of the large ones. For the purpose of probing more deeply into the problem a table was constructed in which the sex-ratios were not grouped. This table is not reproduced here, but from it a diagram was made showing for each litter size the possible sex-ratios and the number of litters occurring with each possible proportion of males. This diagram is reproduced in Fig. 1.

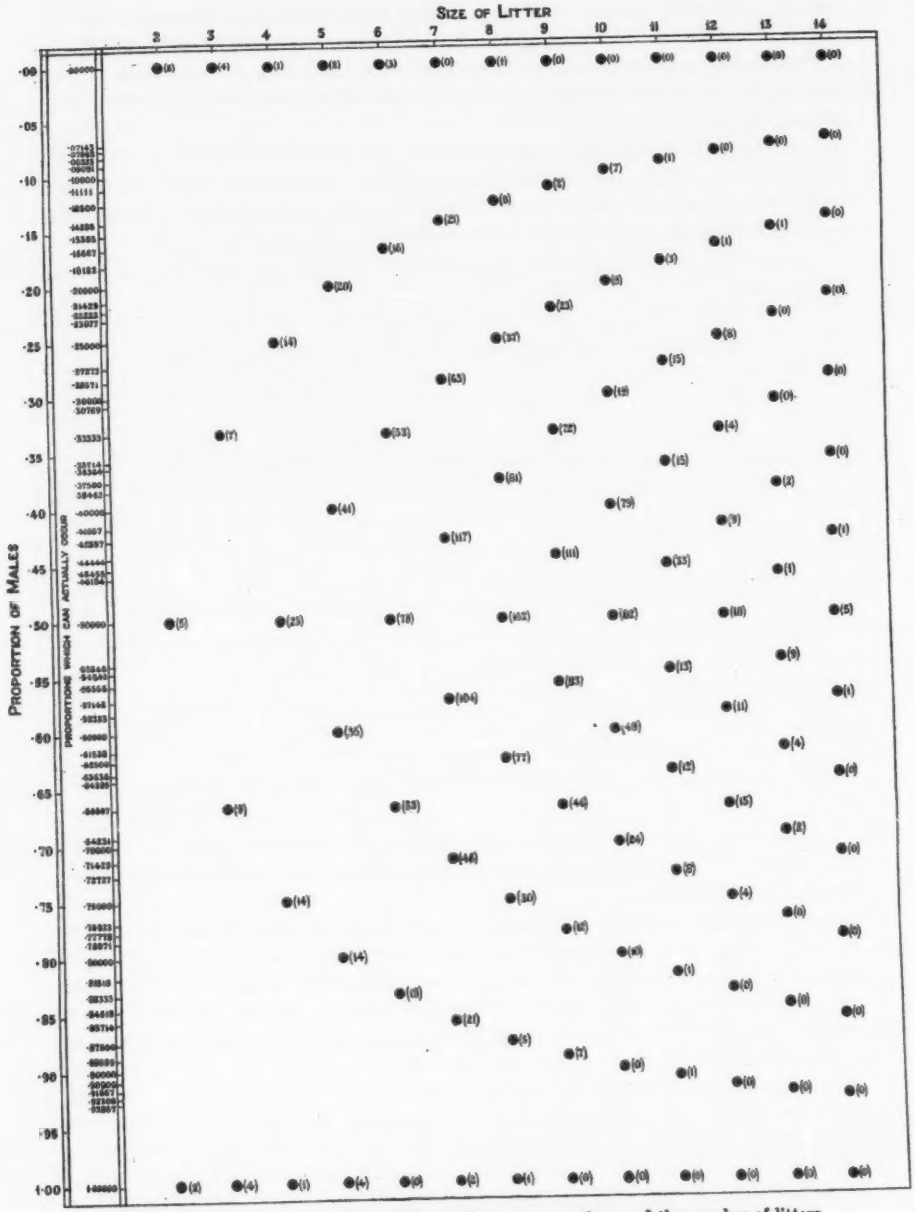


Fig. 1. Showing for each size of litter the possible sex proportions and the number of litters actually occurring at each possibility.

This diagram shows at once the mathematical limitations imposed by the very nature of the data, i.e. the impossibility of getting every sex-ratio in each sized litter. For the two tables as they stand, however, the correlation between size of litter (x) and proportion of males in litter (y) is as follows:

Sex-ratios grouped	Sex-ratios ungrouped
$r = -\cdot0200 \pm \cdot0149$	$r = -\cdot0707 \pm \cdot0149$
$\eta_{xy} = \cdot2337$	$\eta_{xy} = \cdot6155$
$\bar{\eta} = \cdot0667 \pm \cdot0150$	$\bar{\eta} = \cdot1558 \pm \cdot0150$
$\eta_{yx} = \cdot1227$	$\eta_{yx} = \cdot0567$
$\bar{\eta} = \cdot0771 \pm \cdot0150$	$\bar{\eta} = \cdot0771 \pm \cdot0150$

The ungrouped table shows a very small but significant coefficient of correlation, the association being between the larger litters and small proportion of males, but as the regression is obviously not linear no stress can be laid on this result. The correlation ratio of proportion of males in litter on size of litter is not significant since η is less than $\bar{\eta}$, the mean η, when there is no correlation.

The other correlation ratios appear to be significant,and in the usual case actually would be, but in the present instance it is questionable whether this significance is not due to the limitations imposed by the arithmetical treatment, since it is only at the points specified in the diagram that observations can occur. As a consequence of this about 75 per cent. of the arrays for constant proportion of males are such that they can only contain points at their means, i.e. the array can only contain one observation, so that whatever the frequencies in these arrays, η is bound to be large. This is quite obvious in the ungrouped table, but a similar explanation may account for the smaller but still apparently significant η in the grouped table.

It does not, therefore, seem possible to attach any definite biological significance to such a result, and though there is a small significant coefficient of correlation it is not easy to unravel the nature of the association.

For this reason it seemed preferable to calculate the probability frequencies of the various sex combinations on the basis of the mean of the general observations rather than on the individual means given by each size of litter. It will be seen, however, that the binomials resulting from the two bases are not materially different. (Tables IV and V, p. 379.)

(3) *Sex Combinations in Litters.* Attention may now be turned to Table III where the actual and probable frequencies of the different possible sex combinations are given, together with the values of χ^2 and P on Professor Pearson's method (4) of testing goodness of fit. Since litters of 2, 3, 13, and 14 are too few to be representative, the probable frequencies are calculated on the basis of the mean sex-ratio of litters 4–12, according to the terms of the binomial

$$N(q + p)^n,$$

where N is the numbers of litters of the given size, n the number in the litter, and p and q have the fixed values ·4876, ·5124.

In the case of litters of 2, 3, 13, and 14 the smallness of the numbers of litters found causes the value of P to have but little significance, and consideration of these litters will, therefore, be omitted from the following discussion.

TABLE III. *Showing for each size of litter the observed and probable frequencies of the sex combinations, together with the values of the constants for goodness of fit.*

Size of Litter

No. of males in litter	2		3		4		5		6		7		8		9		10		11		12		13		14	
	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.	Ob.	Calc.
0	2	2·3	4	3·2	1	3·7	2	4·1	3	4·0	—	3·5	1	1·9	—	·3	—	·3	—	·07	—	·02	—	·003	—	·0006
1	5	4·5	7	9·2	14	13·9	20	19·5	16	22·8	21	23·1	8	14·5	2	7·2	7	3·3	1	·68	—	·26	—	·039	—	·008
2	2	2·2	9	8·8	23	19·9	41	37·1	53	54·3	63	66·0	37	48·4	23	27·5	8	14·1	3	3·2	1	1·4	1	·23	—	·049
3	—	—	4	2·8	14	12·6	35	35·3	78	68·9	117	104·6	81	92·2	72	61·0	19	35·7	15	9·3	8	4·3	—	·79	—	·19
4	—	—	—	—	1	3·0	14	16·8	53	49·2	104	99·5	162	109·7	101	87·1	79	59·5	15	17·6	4	9·3	—	1·9	—	·49
5	—	—	—	—	—	—	4	3·2	18	18·7	46	56·8	77	83·5	83	82·8	82	67·9	33	23·5	9	14·2	2	3·2	—	·94
6	—	—	—	—	—	—	—	—	—	3·0	21	18·0	30	39·7	46	52·5	48	53·9	13	22·4	18	15·7	1	4·0	1	1·34
7	—	—	—	—	—	—	—	—	—	—	2	2·5	5	10·8	12	21·4	24	29·3	12	15·2	11	12·8	9	3·9	5	1·40
8	—	—	—	—	—	—	—	—	—	—	—	—	1	1·3	7	5·1	10	10·5	8	7·2	15	7·6	4	2·7	1	1·21
9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	·5	—	2·2	1	2·3	4	3·2	2	1·5	—	·77
10	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·2	1	·44	—	·92	—	·55	—	·37
11	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·04	—	·16	—	·14	—	·13
12	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·01	—	·023	—	·030
13	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·002	—	·004
14	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	·0003
Totals	9	9·0	24	24·0	53	53·1	116	116·0	221	220·9	374	374·0	402	402·0	346	345·9	277	276·9	102	101·93	70	69·87	19	18·977	7	6·9319
χ^2=	·6286		1·2857		5·3422		2·1623		6·8308		8·1543		38·3946		15·6360		28·2466		14·3965		17·5642		13·4960		11·7169	
P=	·7303		·7314		·2538		·8277		·3366		·3192		·000006		·0759		·0017		·2112		·1298		·4103		·6390	

A glance at Table III shows that the actual distribution of the litters is very wide of the binomial expectation, the only good fit being found in litters of five. The question immediately arose, therefore, whether this was merely a mathematical result, or whether the observed distribution actually was an improbable one. To find out if the result was brought about by the values given to p and q, i.e. ·4876 and ·5124, the actual distributions in two litters were compared with three sets of binomial probabilities, based respectively on values of p and q given by (1) the mean sex proportions in litters 4–12, (2) the mean sex proportions in all the litters 2–14, and (3) on the sex proportions in the individual litter in question. For this test a litter size with a very low value for P was taken, and a litter size where the fit was not so bad, namely litters of eight and litters of six, where the values for P in Table III are respectively ·000006 and ·3366. In litters of six the three pairs of values for p and q work out at ·4876 and ·5124, ·4884 and ·5116, ·4962 and ·5038. In litters of eight the values are ·4876 and ·5124, ·4884 and ·5116, ·4916 and ·5084.

The results of the comparison of the observed distribution with the various binomials were as follows:

TABLE IV. *The observed Distribution in Litters of* 6 *compared with various Binomial Frequencies.*

No. of males	Observed litters	Calculated, on basis of proportion of males of		
		·4876	·4884	·4962
0	3	4·0	4·0	3·7
1	16	22·8	22·6	21·4
2	53	54·3	54·2	52·2
3	78	68·9	68·9	69·0
4	53	49·2	49·4	51·0
5	18	18·7	18·8	20·2
6	—	3·0	3·0	3·3
Totals	221	220·9	220·9	220·8
$\chi^2=$	—	6·8308	6·7013	6·2917
$P=$	—	·3366	·3494	·3932

TABLE V. *The observed Distribution in Litters of* 8 *compared with various Binomial Frequencies.*

No. of males	Observed litters	Calculated, on basis of proportion of males of		
		·4876	·4884	·4916
0	1	1·9	1·9	1·8
1	8	14·5	14·4	14·2
2	37	48·4	48·0	47·0
3	81	92·2	91·9	90·9
4	162	109·7	109·6	109·7
5	77	83·5	83·8	84·8
6	30	39·7	40·0	41·1
7	5	10·8	10·9	11·2
8	1	1·3	1·3	1·4
Totals	402	402·0	401·8	402·1
$\chi^2=$	—	38·3946	38·4538	39·5521
$P=$	—	·000006	·000006	·000004

In neither case is there appreciable variation in the value of χ^2, or, therefore, in the value of P. As the three pairs of values for p and q used above are the only values which could reasonably be assigned to them, it may be concluded that the discrepancy between observation and probability is not merely a mathematical result brought about by the selection of improper values for p and q, and that this discrepancy is, therefore, a real fact which merits further consideration.

A more detailed examination of Table III at once suggests that the observed data are usually lower than the calculated figures in the "tails" of the distribution and higher in the centre. This impression is confirmed and rendered more precise by a calculation of the standard deviations.

TABLE VI.

No. in litter	Observed standard deviation	Binomial s.d. $\sqrt{npq}$
4	·824	·9999
5	1·069	1·118
6	1·103	1·224
7	1·253	1·323
8	1·194	1·414
9	1·350	1·499
10	1·430	1·581
11	1·688	1·658
12	1·769	1·731

It will be seen that for all litters from four to ten the actual standard deviation is considerably below the binomial value; for the larger litters of eleven and twelve the actual value is exceptionally slightly the greater.

The divergence of observation from calculation may be strikingly brought out by a grouping of Table III. Suppose we group together (*a*) the central classes, taking the actual central class for the even-sized litters of 4, 6, 8, 10 and 12, and the two central classes for the odd-sized litters of 5, 7, 9 and 11, (*c*) the extreme classes as indicated by the heavy rules inserted on Table III, leaving in between (*b*) the intermediate classes. We then have the following summary.

TABLE VII.

	No. of litters		Difference
	Observed	Calculated	
Lower extremes	66	90·4	− 24·4
Lower intermediates	520	551·2	− 31·2
Central classes	890	774·4	+115·6
Upper intermediates	428	477·2	− 49·2
Upper extremes	57	67·4	− 10·4
Totals	1961	1960·6	—

The grouping is of course arbitrary, but any grouping on the same lines would lead to much the same result. The central frequencies are enormously excessive, the remainder entirely in defect. The divergences are obviously beyond the limits of fluctuation of sampling; for the above table as it stands χ^2 is 32·29, n' is 5, and p is only about ·000004, i.e. we could only get real divergence on random sampling once in some 250,000 trials. The excess of the central frequencies for the even-sized litters is perhaps the most remarkable feature of a result which it seems very difficult to explain. The figures run as follows:

4	23	19·9
6	78	68·9
8	162	109·7
10	82	67·9
12	18	15·7
Totals	363	282·1

The observed frequencies are in excess by no less than 81 (80·9) on 282. The total number of these litters is 1023, so that the standard error of the central frequency is

$$\sqrt{\frac{282}{1023} \times \frac{741}{1023} \times 1023} = 14{\cdot}3,$$

i.e. the difference is 5·66 times the standard error.

(4) *Summary.* We may draw the following conclusions:

1. In monotocus animals the deviation of the actual from the probable frequencies of the sex combinations in multiple births is probably accounted for by the prevalence of polyembryony.

2. In normal multiple births, however, this factor can be of little importance, but there seemed to be no information with regard to the probable and actual frequencies of sex combinations in the whole range of litter sizes of any polytocus animal.

3. The analyses here given show that in pigs the mean sex combinations have a greater than probable frequency, while the litters approaching to unisexuality have a less than probable frequency, a finding very hard to interpret.

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THE SOCIAL DISTRIBUTION OF MORTALITY FROM DIFFERENT CAUSES IN ENGLAND AND WALES, 1910–12.

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The following quotation from the *Report of the Inter-Departmental Committee on Physical Deterioration* (1904) shows that some demand exists, for purposes of public utility, for the type of information to which I desire now to direct your attention. After recommending that "a coordinate record should be kept concerning the health of those sections of the people whose sickness is treated at the public expense" the Report goes on to say, "of great value for some purposes are the Registrar-General's statistics concerning the mortality of the whole population without distinction of social status; but, for the reason that no particulars are given on the latter point, those statistics are of little value in the present connexion."

Perhaps the statistics then at the Committee's disposal were of more potential value than they realized, even for the purpose they had in mind, owing to the possibility of combining the returns for separate occupations into groups more or less representative of social status, but however this may be, the task has now been definitely undertaken by the Registrar-General himself.

The material for the attempt now made to investigate the social distribution of fatal disease is provided by the *Report on Occupational Mortality during* 1910–12, recently issued as Part IV of the Registrar-General's Decennial Supplement. This includes for the first time an attempted grading of the male working population into eight social classes as determined by occupation. This grading was first applied to comparisons of infant mortality in the Registrar-General's Report for 1911, and particulars regarding it will be found there. It is unsatisfactory in many ways, as there pointed out, chiefly because of defects in the occupational classification on which it is based. Social status can broadly be deduced from a genuinely occupational classification, but not from an industrial, where all grades of workers, master and man, skilled and unskilled, are grouped together for each industry; and it is chiefly because the "occupational" classification used at the 1911 and preceding censuses is largely industrial in fact that the social grading based on it is defective. Much better results should be attainable in future, for a determined attempt was made to purge the occupational classification of its industrial taint in 1921, accompanied by the introduction of an entirely independent industrial tabulation. It is extraordinary, however, how industry, though expelled with a fork, keeps

none the less returning to the classification of occupations, and I am not sure that even now the assistance of the Board of Trade, Home Office, and Ministry of Labour has enabled us entirely to rid the occupational tabulation of the industrial element.

But even the imperfect social grading based on the 1911 classification gives results which appear, broadly considered, to be fairly satisfactory. This has been established by its application to infant mortality and to fertility. Eight social classes have been distinguished, of which I represents as far as possible the middle and upper classes, liberally interpreted, III skilled labour, and V unskilled labour. II includes occupations which as classified comprised large elements both of I and III, and IV is similarly intermediate between III and V. The remaining three groups represent respectively textile workers, miners, and agricultural workers, whose importance seemed such as to warrant evasion of the difficulty of deciding between skilled and unskilled for each of the occupations comprised by distinguishing each group as a whole.

It follows from what has been said that if the attempted social grading has been in the main successful, and if infant mortality and fertility increase downwards along the social scale, the values obtained for each should increase with some regularity from Class I to Class V; and this has proved to be the case, both in regard to births and infant deaths registered in 1911 and to the records of children born and dead to married couples obtained at the 1911 census. Infant mortality in 1911 was 76·4 for Class I (and much less for middle class occupations on a stricter basis of selection), 106·4 for Class II, 112·7 for III, 121·5 for IV, and 152·5 for V. For the same year legitimate births per 1000 married males under 55 years of age were I 119, II 132, III 153, IV 158, and V 213. The census data of families, similarly treated, give very similar results, in regard both to fertility and child mortality, though these are not so readily quotable, as duration of marriage has to be taken into account.

It appears therefore that the scheme of social grading used in the Report under consideration is, though faulty in detail, adequate for the purpose; and therefore that if adult male mortality does not correspond in its variations with social position as closely as do child mortality and fertility this must be due either to special imperfection in the data or to actual want of correspondence between mortality and social status.

I am inclined to think, for reasons to be stated shortly, that the former cause goes far to explain the fact that correspondence in this case is less precise; but I should like to say that the following remarks are based only upon a somewhat hasty examination of the data. To go at all fully into the matter would involve much labour which I must disclaim having bestowed upon it. All I can pretend to do at present is to point out certain features of, as it seems to me, considerable interest, which stand out on first examination of this part of the Report, while leaving many points unexplained which seem to call for explanation, or at most hazarding a guess as to their meaning, in the absence of a demonstrable solution.

The Report records the deaths registered during 1910–12 for each occupational

class dealt with, distinguishing 8 groups of age and 27 of cause. Females are not dealt with, because the practical difficulties of occupational classification of deaths in their case have so far proved insurmountable. For the four decennial age groups covering the 40 years 25—65 death-rates from each cause dealt with are shown in the form of "deaths in standard population," the standard population being the number of males aged 25—65 in 1901 amongst whom 1000 deaths occurred in 1900–02. This number is 71,005, divided into four age groups in the same proportions as the 1901 population. By adding together the deaths from any cause resulting to these four age groups from the mortalities experienced at those ages in 1910–12 "comparative mortality figures" are obtained, which, it will be seen, are really standardized death-rates obtained by the direct method of standardization, but expressed per 71,005 instead of per 1000 living. Like other standardized death-rates they eliminate differences in mortality resulting merely from variations in age between the populations compared. In making comparisons between the social classes the figures quoted will chiefly be the percentage ratio of "deaths in standard population" for each class from the various causes to the corresponding rates for the occupied and retired of all classes, these ratios of course being the same as for the death-rates (per 1000) themselves, whether for any of the four age groups, or, standardized, for all four jointly.

Deaths from All Causes.

The social classes may first be compared in regard to mortality from all causes. The first of the tables appended shows that at 25—65 the ratios for the five graded classes are 88, 94, 96, 93, and 142. There is thus surprisingly little difference between Classes I and IV, and the latter actually returns a lower mortality than either II or III. The great excess of 42 % for unskilled labour (Class V) is probably in large measure genuine. Excess is to be expected, and large excess of infant mortality for this class has been already noted. But it seems unlikely to be entirely genuine, in view of the closeness of the other four classes. This suspicion is confirmed by the fact that the cancer ratio for this class is also very high, 131, whereas for miners and agricultural labourers, in whose case one would expect diagnosis to be on a somewhat similar level of efficiency, it is very low (78 and 68). Apart from the variations quoted, cancer mortality varies little with class, and this and other circumstances induce suspicion that the Class V excess may be largely due to that class being saddled with more than its fair share of deaths.

It has often been pointed out that the deaths being obtained by registration and the population by census count the two do not necessarily correspond. Men suffering from chronic disease tend to seek lighter and necessarily unskilled occupations, so that Class V must receive many recruits with but a few months to live. But in addition to this consideration another factor has probably transferred to Class V some of the mortality belonging to Class IV. For reasons discussed in the *Report on Occupational Mortality*, death-rates for general labourers as tabulated are much exaggerated, and those for specialized labourers understated, a table showing the large extent to which this has occurred being included in the Report. All general

labourers are assignable to Class V, but many specialized labourers are necessarily included in Class IV owing to inability to differentiate them from skilled workers in the same industries. Consequently the mortality of Class IV must be understated at the expense of Class V.

The death-rate, for instance, of woodsawyers, who are included in Class IV, is returned as only 67 % of the general average. It seems certain that this must be due in great measure to inclusion with this group in the census of many labourers who happened at census date to be working in saw-yards but whose occupation would be correctly stated in registration as general labourer.

Retransfer of even a small portion of its excess mortality from Class V to Class IV suffices to make the increase of mortality with descent in the social scale continuous, as in the case of infant mortality, where, owing to the fact that all the data (births and deaths) are derived from registration, this trouble does not arise. What the extent of the transfer should be it is of course impossible to say, but I am disposed to think that it would be a very moderate assumption to suppose that 10 % of the mortality credited to Class V properly belongs to Class IV. The effect of this assumption may be seen from the following statement of its results upon the mortality ratios of the five graded classes as given in Table I.

	Ratios as returned					Modified Ratios				
	I	II	III	IV	V	I	II	III	IV	V
25—	88	98	89	96	146	88	98	89	109	131
35—	85	97	92	93	151	85	97	92	107	136
45—	87	94	97	92	144	87	94	97	106	130
55—	90	91	100	93	133	90	91	100	106	120
25—65	88	94	96	93	142	88	94	96	107	128

The second series seems much more reasonable than the first, and as it is certain that a considerable modification of the kind is required, whatever its precise extent may be, I shall assume the Class IV and Class V rates from each cause discussed liable to alteration on this scale, such series as that just quoted being regarded as regularly progressive without reference in every case to the modification which makes them appear so.

The ratio for textile workers is 101, for miners 94, and for agricultural labourers 59. The extraordinarily low rates of the latter class from practically every cause dealt with form one of the most remarkable features of the tabulation. Moreover their ratio improves from 68 % at 25—35 to 56 % at 45—55 (58 % at 55—65) showing the progressive effect of their healthy conditions of life. The only other group showing definite improvement with age is Class II, whose ratios decrease from 98 at 25—35 to 91 at 55—65, whereas Class III (skilled labour) moves in the opposite direction, from 89 to 100, Class VI (textile workers) from 90 to 116 and Class VII (miners) from 94 to 101. Both the two latter groups show some improvement at 35—45 as compared with 25—35, but after that the increase from 86 to 116 for textile workers and from 88 to 101 for miners must, I think, be due to delayed effect of their conditions of work. The same statement may apply to Class III, most of the great artisan occupations belonging to which imply indoor, and often undesirable, conditions of work.

The rates for both Classes VI and VII show further relative increase at ages over 65, whereas the middle class never shows to so much advantage as after 75 is passed (72 % of normal).

Before leaving the comparison of total mortalities it may be well to point out that the Class I proportion of 88 % is much higher than it would otherwise be on account of the inclusion in this class of certain large occupations with comparatively high mortality, particularly commercial clerks, which might perhaps have been grouped with Class II as by no means exclusively middle class. Commercial clerks alone form 28 % of Class I, and have a mortality ratio of 109 (per cent. of average), whereas four occupations selected out of those available as most definitely assignable in their entirety to Class I—clergymen, barristers and solicitors, medical practitioners, and bank officials, totalling 103,000 men, or about 8 % of Class I—have a ratio of 70, that for all Class I being 88.

It has been necessary to say so much as to mortality from all causes in order to provide a background against which the significance of class variations in mortality from particular causes may be appreciated. These will now be considered.

Phthisis. The proportion of the total deaths (in the standard population) ascribed to phthisis was 17·9 % (Table II), varying from 11·7 in the case of Class VII (miners) to 20·4 in that of Class III. It was by far the most important single cause of death in all of the classes so far as disease is concerned, being exceeded only by accident in the case of miners.

Amongst the five graded classes mortality ranged from a minimum of 86 % of average in Class I to a maximum of 152 % in Class V, but the sequence was not regular, the apparent superiority of Class IV (92 %) over Class III (109 %) being much greater than from all causes. The Class V proportion, 152 %, is considerably worse than that from all causes, 142 %, and it is chiefly on it that social grading of the unmodified rates of phthisis mortality depends, for the difference between Class I and Class IV is only that between 86 and 92. Modification in the sense indicated above, however, furnishes a much more reasonable series, and one better in keeping with the well-established association between poverty and phthisis. It may well be that in this case, as in that of other chronic disabling diseases, a greater reduction of Class V mortality is required than in dealing with deaths from all causes, for it is in these cases that health considerations so often demand a change to lighter work. This of course would imply transfer to other classes as well as IV, whereas the discrepancy between the census and registration classifications affects Classes IV and V alone. By far the lowest proportions are those for miners (61) and agricultural labourers (50).

Dealing with the four age groups separately we find that at 25—35 there is no indication whatever of increase down the social scale until we come to Class V. The Class I rate is in this case actually 6 % above average, and that returned for Class IV 9 % below it, so it seems that in early adult life there is little association between social status and phthisis mortality. The Class V excess of 38 % is less

TABLE I.

Death-Rates from various Causes in Social Classes per cent. of those for all Occupied and Retired Males of the Ages distinguished.*

(Mortality at 25—65 standardized.)

	Age	Social Class							
		I	II	III	IV	V	VI	VII	VIII
All Causes	25—35	88	98	89	96	146	90	94	68
	35—45	85	97	92	93	151	86	88	62
	45—55	87	94	97	92	144	100	88	56
	55—65	90	91	100	93	133	116	101	58
	25—65	88	94	96	93	142	101	94	59
Tuberculosis	25—35	105	98	103	92	137	93	54	61
	35—45	88	88	108	90	152	85	60	58
	45—55	71	81	111	96	157	90	68	45
	55—65	66	73	115	92	163	112	78	42
	25—65	86	87	108	92	150	93	63	54
Diseases of the Nervous System	25—35	85	99	84	97	164	73	75	80
	35—45	104	95	94	94	147	79	75	56
	45—55	103	100	99	90	135	115	73	49
	55—65	99	91	107	93	121	123	99	50
	25—65	100	95	100	92	134	108	85	54
Diseases of the Circulatory System	25—35	82	101	80	99	163	106	79	57
	35—45	76	90	86	105	165	78	79	60
	45—55	83	92	96	92	147	107	92	63
	55—65	97	94	99	94	125	120	100	62
	25—65	89	94	95	96	141	108	93	62
Diseases of the Respiratory System	25—35	66	92	83	100	174	93	88	58
	35—45	64	94	85	91	177	88	88	61
	45—55	61	79	90	95	179	100	107	50
	55—65	60	75	96	97	163	116	135	50
	25—65	62	82	90	96	171	103	113	53
Diseases of the Digestive System	25—35	101	107	87	103	120	116	88	74
	35—45	119	135	85	86	122	99	67	58
	45—55	124	129	91	86	114	101	69	51
	55—65	124	115	93	89	111	106	88	58
	25—65	120	122	90	89	115	104	78	58
Diseases of the Urinary System	25—35	78	96	98	98	150	113	63	70
	35—45	89	104	105	95	129	103	77	58
	45—55	107	104	108	91	123	121	55	40
	55—65	111	107	109	86	113	126	75	45
	25—65	103	104	107	90	123	119	68	48
Phthisis	25—35	106	98	105	91	138	92	52	59
	35—45	88	88	108	90	154	82	58	52
	45—55	70	81	112	96	159	89	66	43
	55—65	65	72	115	92	166	105	78	39
	25—65	86	87	109	92	152	90	61	50
Cancer	25—35	103	90	100	100	138	76	72	83
	35—45	97	95	95	92	138	103	80	66
	45—55	100	88	101	100	133	93	79	63
	55—65	98	92	103	95	128	108	78	71
	25—65	99	91	101	96	131	102	78	68
Diabetes	25—35	127	107	107	87	113	*93*	87	*60*
	35—45	129	118	82	94	94	*88*	82	*71*
	45—55	126	148	96	83	87	91	57	*39*
	55—65	171	160	88	86	64	131	50	33
	25—65	146	141	92	87	82	108	63	45
Tabes Dorsalis and General Paralysis of the Insane	25—35	71	81	90	110	195	*62*	57	*38*
	35—45	111	92	92	98	151	60	71	24
	45—55	131	102	89	89	140	76	66	32
	55—65	131	97	100	106	117	89	74	46
	25—65	117	95	92	98	146	71	69	33

* No ratios are shown for death-rates based on less than ten deaths; and where the basis is less than twenty deaths the ratios are printed in italics.

TABLE I (*continued*).

	Age	Social Class							
		I	II	III	IV	V	VI	VII	VIII
Cerebral Haemorrhage, Apoplexy	25—35	90	120	60	110	140	—	90	*30*
	35—45	85	109	100	85	138	115	71	47
	45—55	92	99	112	91	124	142	63	42
	55—65	96	91	109	94	117	133	103	46
	25—65	94	96	108	93	121	132	88	46
Valvular Disease of the Heart	25—35	78	100	83	93	165	108	95	45
	35—45	74	88	90	101	167	75	86	65
	45—55	76	84	99	93	155	101	94	67
	55—65	84	91	97	95	135	110	106	64
	25—65	79	89	95	95	149	101	98	63
Angina Pectoris and Arterio-Sclerosis	25—35	—	—	—	*200*	—	—	—	—
	35—45	*80*	120	100	100	180	—	*80*	—
	45—55	158	108	92	96	108	113	63	*33*
	55—65	170	106	100	89	89	127	79	37
	25—65	161	106	98	92	99	120	75	37
Aneurysm	25—35	*60*	100	60	200	140	—	—	—
	35—45	53	79	79	153	180	—	47	*47*
	45—55	64	96	96	112	164	*48*	76	*24*
	55—65	100	100	95	105	130	*70*	70	75
	25—65	71	93	88	128	158	46	62	51
Bronchitis	25—35	*36*	86	93	86	207	*114*	64	*36*
	35—45	41	76	84	103	208	111	95	*19*
	45—55	42	66	93	97	194	117	111	42
	55—65	40	66	98	99	174	128	149	39
	25—65	41	68	95	98	184	123	131	37
Pneumonia	25—35	67	93	81	103	173	93	90	56
	35—45	69	97	84	88	177	88	81	58
	45—55	68	82	86	93	182	89	100	54
	55—65	77	82	90	93	161	105	114	59
	25—65	71	87	86	94	173	94	98	59
Hernia	25—35	—	*100*	*100*	*100*	*150*	—	—	—
	35—45	—	133	100	*67*	167	—	—	—
	45—55	*67*	100	83	117	167	—	*100*	*100*
	55—65	73	100	100	100	136	*82*	155	*73*
	25—65	64	105	95	100	150	77	123	100
Cirrhosis of the Liver	25—35	*60*	180	80	100	160	—	—	—
	35—45	133	196	71	67	121	*42*	50	*33*
	45—55	139	184	82	80	98	82	49	24
	55—65	149	153	82	80	100	95	58	40
	25—65	139	173	80	78	105	80	53	33
Other Diseases of the Digestive System	25—35	106	102	87	103	116	119	92	76
	35—45	117	114	89	93	120	120	71	60
	45—55	120	102	97	87	119	112	77	61
	55—65	117	98	97	93	114	114	95	66
	25—65	116	103	93	93	117	115	85	65
Bright's Disease	25—35	86	100	97	97	148	128	59	66
	35—45	89	107	105	96	122	107	67	55
	45—55	110	107	112	89	118	130	47	37
	55—65	119	110	112	82	107	126	71	40
	25—65	108	107	110	88	116	124	62	44
Suicide	25—35	124	129	68	108	129	84	61	68
	35—45	98	131	79	106	108	113	56	77
	45—55	119	133	96	79	110	117	56	65
	55—65	83	115	104	88	119	148	73	54
	25—65	105	127	88	94	116	117	61	66
Accident	25—35	41	73	60	91	156	45	284	70
	35—45	51	87	62	82	164	40	251	66
	45—55	50	87	70	81	175	43	229	74
	55—65	52	80	82	82	165	73	212	81
	25—65	48	82	68	84	165	49	246	72

than that from all causes (46). The proportion for miners at this age (52 %) is actually below that for agricultural labourers (59), but with increasing age the miners' proportions increase regularly to 78 % at 55—65, while those for agricultural labourers decrease regularly to 39 % at the same age. The miners' figure further increases to 88 % at 65—75, and in this respect their experience in regard to phthisis resembles that from other respiratory diseases, young miners returning low rates which steadily increase as life advances up to and beyond 65. But in these cases, and especially in that of bronchitis, the rates for elderly miners are far in excess of the general average. Evidently whatever the cause may be which confers immunity to respiratory disease upon miners its operation is rapidly established, but does not provide a permanent protection throughout the whole of life against the evils of exposure to dust.

As age increases the Class I proportion to total mortality gradually falls from 106 % at 25—35 to 65 % at 55—65, that of Class II from 98 to 72, and of Class VIII, as already stated, from 59 to 39; while Class III rises steadily from 105 to 115, Class V from 138 to 166, and Class VII (miners) from 52 to 78 (and 88 at 65—75). It follows from this that phthisis mortality occurs early in life in Classes I, II and VIII and late in V and VII, as may be seen from the following statement of phthisis deaths in standard population as quoted in the Report in the form of deaths at each age per 1000 deaths at 25—65.

	All classes	I	II	III	IV	V	VI	VII	VIII
25—	312	385	351	299	309	283	318	265	368
35—	300	307	303	298	293	304	273	285	313
45—	239	194	223	246	249	250	236	260	203
55—	149	114	123	157	149	163	173	190	116
25—65	1000	1000	1000	1000	1000	1000	1000	1000	1000

Class I provides the highest proportion in the series at 25—35, and the lowest at 55—65; and Class VII (miners) provides the lowest proportion at 25—35 and the highest at 55—65. The five graded classes show a regular sequence in each case, except for interruption at Class IV, which descends from the Class I maximum at 25—35, and ascends from the Class I minimum at 55—65. It thus appears that the variation of age at death from phthisis in this country with geographical position, on which much stress has recently been laid, is accompanied by similar variation with social position, a fact which seems to me to increase the difficulty of attributing the geographical differences to variations in the infecting organism.

It should be stated that as with all causes the Class I proportion of 86 % from phthisis would be very much lower under a more restricted interpretation of the term "middle class." For the four occupations (professions and banking) already used for this purpose the ratio is only 42 instead of 86 %, the difference being accounted for by high rates in other cases, as commercial clerks, 136, and law clerks, 130 % of average. This excess is much greater than from all causes (commercial clerks 109 %). It seems not unlikely that this occupation may receive recruits whose health, already impaired by tubercle, forbids their continuance in some more strenuous calling.

Cancer. The rates for Classes I—IV and for Class VI (textile workers) approximate closely to the general average, those for Class II being generally somewhat lower than the others. The proportion for Class VII is only 78 % and for VIII 68, the rates in these two cases being consistently low at all ages. That for Class V, 131, is alone in any considerable excess, but in view of what has already been said about the data for labourers it will be seen that this figure must overstate their excess. Transfer of 10 % of the Class V mortality to Class IV brings the rates for both IV and V into definite though moderate excess, and that although it might be expected that more cases would be missed in this class than others. If this consideration is of importance the mortality of Class I may not be so close to average as it appears on the surface, since the proportion of cases missed should be at a minimum for it; and this surmise derives some support from examination of the rates for the four selected occupations. Those for lawyers and doctors are slightly below average, those for bankers considerably, and for clergymen very much so, and yet it is just here that one would expect fewest cases to be missed. Commercial clerks have a high rate, 28 % in excess of average, so here again is possible evidence of recruitment by invalidism. This is stronger still in the case of the Class V occupation messengers, their excess being no less than 51 %. Generally speaking then, there seems to be evidence of moderate increase of cancer mortality down the social scale, but with marked exceptions in favour of the very specially situated occupations of mining and agriculture.

Diabetes exhibits a social distribution so striking and so closely in accord with accepted opinion on the subject that little doubt can be entertained as to its significance. For the five graded classes the ratio for the whole period 25—65 falls from a maximum of 146 for Class I to a minimum for Class V—and that notwithstanding overstatement, common to this and all other causes—of 82. The ratio for textile workers is 108, for miners 63, and for agricultural labourers only 45. These differences moreover increase largely, on the whole, with age. At 25—35 the range is from 60 for agricultural labourers to 127 for Class I, but at 55—65 it is from 33 for the former to 171 for the latter. The conditions making for high mortality in the one case and low in the other are evidently cumulative in their effect.

Syphilis. The returns of deaths certified as due directly to this disease have not been tabulated, but a better guide to its social distribution may be obtained from the larger and less prejudiced figures of deaths from tabes dorsalis and general paralysis of the insane, both of which diseases are now accepted as implying infection with syphilis. Their mortality is almost double that of diabetes, so the numerical basis is respectable, and in this case it may be assumed that diagnosis is equally good for all classes, for the poorest man suffering from either disease is almost bound, I take it, before his death to come under specially skilled observation. Certainly this is true of general paralysis of the insane, which caused three-fourths of the joint mortality.

Table I (p. 387) shows that after transfer, as in other cases, of some portion of the Class V mortality to Class IV, the death-rate appears to be in excess at the two extremes

of the social scale, but very low for the three special groups, and particularly for agriculture. For some reason the Class I ratio increases with age, from 71 at 25—35 to 131 at 45—65, while that for Class V decreases from 195 at 25—35 to 117 at 55—65. Either infection occurs later in life in Class I or the interval between infection and death is greater. The relative immunity of the three special groups is curious. I think it would be very rash to attribute it to greater morality, but I cannot help coupling it with another circumstance relating to the same three groups. In fertility statistics ante-nuptial conception is found to involve overstatement of the duration of marriage during its first year or two, and in 1911 this overstatement was very much less for Classes VI—VIII than for Classes I—V. It was at a maximum for Class V, so it will be seen that there is close general correspondence between these two sets of entirely independent data. None the less, I greatly hesitate to accept the moral pinnacle which might be inferred from them for the classes in question.

Cerebral Haemorrhage. For the total population compared (25—65) proportions gradually increase down the social scale from 94 for Class I to 121 for Class V. The highest ratio however is 134 for textile workers, those for miners and agricultural labourers being 88 and 46. Practically the same distribution of mortality applies at 45—55 and 55—65, at which alone the death-rate is serious, the small rates at the two earlier ages being much more erratic in their distribution. The high rates for textile workers under this heading are repeated in the case of deaths classified to angina pectoris and arterio-sclerosis, so it would seem that arterial degeneration and its cerebral consequences are for some reason particularly common in these occupations.

Nervous Diseases generally. Classes I—IV depart little from the average, but Class V returns a ratio of 134, VI coming next with 108. Miners and agricultural labourers are lowest, with ratios of 85 and 54. As in many other cases the position of miners is much worse (99) at 55—65 than at earlier ages (73—5) whereas that of agriculturists improves with advancing age. The same contrast applies between Classes V and VI, the position of the former improving from 164 at 25—35 to 121 % of average at 55—65, while that of the latter deteriorates from 73 to 123.

Valvular Disease of the Heart. This is in one respect an unsatisfactory heading, inasmuch as even now almost a quarter of the deaths from heart disease are merely returned as such, without differentiation of the form of disease, and no doubt the proportion was larger in 1910–12. It follows that where precision of certification is greatest the death-rate from valvular disease must tend, *ceteris paribus*, to be highest. Some degree of apparent excess for Class I would therefore seem quite compatible with actual uniformity for all classes. But instead of this the ratios increase amongst the five graded classes from 79 for Class I to 149 for Class V. It seems clear therefore that valvular disease is more fatal in the lower ranks of society. This is probably what might be expected for two reasons: (1) the importance of care during and after an attack of acute rheumatism if permanent damage to the heart is to be avoided, and (2) the greater liability of the lower ranks to injury by strain, which, as will be seen, is well brought out by the returns for aneurysm and hernia.

The mortality ratio from this cause of the more strictly defined middle class may be gathered from the figures for the same four occupations which have already been used for this purpose. They are—clergymen 44, lawyers 31, doctors 51 and bankers 62 %—all much below the Class I average of 79. And once more it is commercial clerks (105) who are largely responsible for bringing the class figure even up to 79.

Angina Pectoris and Arterio-Sclerosis. These two allied causes have been grouped together, but about three-quarters of the deaths were ascribed to the latter. Attention has already been drawn to the excess mortality from this cause of textile workers (120). But their rates are far exceeded by those for Class I (ratio at all ages 161), the Class II ratio being 106. It is much to be doubted however whether any significance attaches to this at first sight impressive Class I excess. In 1910–12 the vogue of arterio-sclerosis in death certification was increasing by leaps and bounds. In such a case it seems reasonable to assume that the fashionable term makes its way most rapidly in fashionable practice, and I for one am quite prepared to accept the 61 % excess for Class I as probably due entirely to this cause, and to regard the 99 % mortality of Class V as correspondingly understated in comparison.

Aneurysm. Amongst the five graded classes mortality increases from a ratio of 71 for Class I to 158 for Class V. This distribution differs entirely from that of the other sequels of syphilis dealt with, tabes dorsalis and general paralysis of the insane, in that their high Class I figure is replaced by a low one. Possibly, as the disease is one partially amenable to treatment, superior medical attention in the case of Class I may account for the contrast, but it is also possible that liability to strain may explain it, occupations involving heavy labour generally returning a high aneurysm mortality. However this may be it is interesting to note that for Classes VI—VIII the mortality is exceedingly low, the ratios being 46, 62 and 51 %. These are even lower than the tabes dorsalis and general paralysis ratios already quoted, 71, 69 and 33, and so support the suggestion of the latter that for some reason these classes suffer comparatively little from syphilis. If it were not so one might expect the miner's ratio to be much higher than 62, in view of the liability to strain involved by his calling. This is borne out by a high miners' ratio from hernia (123). Agricultural labourers also return a ratio from hernia twice that from aneurysm, whereas for textile workers, who are presumably little subject to strain, the hernia ratio is only 77. Where the syphilis rate is low the aneurysm rate is low also, strain or no strain, but where syphilis paves the way for aneurysm the figures suggest that strain is a material consideration.

It should be stated that the basis of fact is smaller for aneurysm than for any cause previously dealt with, the deaths tabulated for all classes numbering 2319.

Diseases of the Circulatory System generally. Allowing for understatement of Class IV deaths, as in other cases, Table I (p. 387) shows moderate increase from the middle class downwards to unskilled labour. For our four strictly middle class occupations jointly the ratio is 81, so the degree of range is very considerable, the Class V ratio, as returned, being 141.

Bronchitis. In this case the range is extreme, from 41 for Class I to 184 for Class V. The association of poverty with respiratory disease generally is most pronounced, and it appears at its maximum in the case of bronchitis. For the four selected middle class occupations the ratio is only 15 %, no deaths from bronchitis at all being registered for either clergymen or bankers under the age of 55, while the ratio for the whole of Class I is raised, as usual, by a comparatively high figure for commercial clerks (74 % of average). In any repetition of this work it is perfectly clear that our present Class I must be subdivided, even though the numbers in the upper division be small, for as it stands it is far too heterogeneous from the mortality point of view.

It is probable that some deaths may be loosely ascribed to bronchitis in working-class practice which are not so returned for Class I. The extent to which this occurs in the case of elderly phthisical patients has been much discussed without, so far as I have been able to see, the establishment of any very definite conclusion. But I doubt whether this can go very far to account for the enormous excess of bronchitis mortality amongst the poor. The exposure of certain occupations to conditions of labour leading to high bronchitis mortality may go further towards providing the explanation. The highest rates are returned by the following occupations, ratios being as stated in each case: potters 450, cotton strippers and grinders 395, zinc makers 339, tailors in Bethnal Green and Stepney 337, cotton blow-room workers 332, sandstone masons 305, costermongers 274. It will be seen that a definite dust risk attaches to most of these occupations, though not to all.

The cotton operatives quoted are peculiar in that their enormous respiratory mortality is accompanied only by an average phthisis rate, and that with them as with the East End tailors the bronchitis rate is far higher than the pneumonia. Agricultural occupations generally have very small bronchitis mortalities, e.g. those for farmers and gardeners as well as agricultural labourers, but open air occupation is no protection, as evidenced by the costers already quoted, and by high rates for bargemen, carmen and others.

But when full allowance has been made for special occupational risks, including that of missed phthisis, there would almost seem to be some association between poverty *per se* and bronchitis.

Textile workers as a whole show moderate excess (11—28 %) at all ages, but the rates for miners are interesting as exhibiting in marked degree the same good start in early life followed by rapid deterioration later on which characterizes this occupation in regard to tubercle and to nervous circulatory and respiratory diseases in general. From 64 % at 25—35 the ratio rises to 149 at 55—65 and to 184 at 65—75.

Pneumonia (all forms jointly). Here again the Class I—V range is quite regular, but only from 71 % for Class I to 173 for Class V as against 41 to 184 for bronchitis. The ratio for the four selected middle class occupations is 58 (the mortality of medical men from this cause being practically the same as for all males, both in 1900–2 and in 1910–12)—a very different story from bronchitis—and for agricul-

tural labourers it is 59. It is slightly below average for both textile workers and miners, but for the former it rises from a minimum of 88 at 35—45 to 105 at 55—65, and for the latter from 81 to 114 at the same ages, and to 128 at 65—75.

The range for pneumonia is less than for bronchitis at the higher as well as at the lower end of the mortality scale, the highest ratios worth quoting which I have noted being for barmen 276, costermongers 210, and dock labourers 209. High pneumonia rates are generally, but not always, accompanied by high phthisis rates.

Respiratory Diseases in general. Pneumonia and bronchitis are so preponderant here, causing 88 % of the total deaths dealt with, that the joint figures practically repeat what has already been noted for the two main ingredients. The Class I—V range is from 62 to 171 %, proportions for VI, VII and VIII being 103, 113 and 53 %. The ratio for textile workers rises from 88 at 35—45 to 116 at 55—65, and that for miners from 88 at 25—45 to 135 at 55—65 and 166 at 65—75, showing the slow but cumulative effect of their conditions of labour. For the four selected occupations the ratio is 45. The pneumonia ratio for medical practitioners of almost 100 % brings their total respiratory proportion to 70 %, whereas for clergymen it is 35, for lawyers 54, and for bankers only 32 %.

Cirrhosis of the Liver (4484 deaths). Here the social distribution is very different from that of respiratory disease, Class II coming first with 173 % and Class I second with 139. On the whole the record agrees well with that of deaths ascribed directly to alcoholism, which has not been quoted owing to the small numbers on which it is based. Here also Class II (174) comes first, and Class I is above average, but in this case Class V exceeds it. And here again the rates for miners and agricultural labourers are exceedingly low.

On examination of separate occupations the Class II excess is seen to be caused by high rates for various shopkeeping occupations, especially retailers of drink. The ratio for publicans is 900, for butchers 262, fishmongers 192 and tobacconists 192, all these being Class II occupations. For shopkeepers as a whole the ratio is 146. Next to the imposing figure for publicans, but at a long interval, come other occupations connected with the manufacture or sale of alcoholic drinks—barmen 454, cellarmen and beer bottlers 354, brewers 300, and waiters 185. After these some of the highest rates are returned by Class I occupations, though here the clergy, with a ratio of 23 %, must be absolved. But the proportion for chemists and druggists (shopkeepers, but in Class I) is 223, artists, etc. 200, lawyers 185, lawyers' clerks 177, and doctors 169. For commercial travellers the ratio, 146, is hardly perhaps as high as might have been expected. The only other ratios to compare with those quoted (omitting numerically small occupations) are for hairdressers, 215, and costermongers, the only Class V occupation in this list, 177. Amongst the lowest ratios recorded may be mentioned tramway workers 23, iron miners 23, motor drivers 15, and gamekeepers 15 %. No doubt drinking drivers have to go. On the whole the impression is left that it is not so much the man who receives a weekly wage who poisons himself with alcohol as the man who has access day in day out to money in the till or drink from the tap.

Diseases of the Digestive System generally form a very striking contrast to those of the respiratory system in their social distribution. The highest ratios are 122 for Class II and 120 for Class I; III—V are all considerably lower when allowance is made for transfer of mortality from IV to V. On extracting the list of occupations with highest rates this is found to be so much influenced by cirrhosis of the liver, publicans and barmen coming first, that it is almost a repetition of the cirrhosis list. For this reason "other" has been substituted for "all" digestive diseases, i.e. all other than cirrhosis and hernia. Class I (116) now really occupies first place, for the Class V ratio of 117 % must of course be exaggerated, as in other cases. The textile ratio is 115, while those for miners and agricultural labourers are again low (85 and 65). Reliable ratios for separate occupations range from that for doctors (176) downwards. Verily the physician has not yet learnt to heal himself. Three of the four selected middle class occupations in fact return high ratios, that for lawyers being 156 and for clergymen 124, but the bankers' rate is only 79. Other high ratios are for publicans 168, barmen 165, potters 159, wool and worsted workers 144, textile dyers and finishers 141, musicians 141, and dairymen 138. Evidently alcohol has some influence here also, and an abundant menu seems on the whole to involve more risk than a scanty one. How far the high rates for doctors, lawyers and clergymen may be due, if at all, to superiority of the medical skill available to them is a matter on which I would rather solicit than hazard an opinion.

Bright's Disease. When allowance is made for transfer between IV and V the rates for the five graded classes are all very much the same, but the Class I ratios increase, and those for Class V decrease, regularly with advancing age. Class VI (textile workers) records the highest ratio (124), and miners (62) and agricultural labourers (44) much the lowest.

Suicide. Class II (127) returns the highest ratio, that for shopkeepers as a whole being 121. Textile workers come next with 117, their ratios showing continuous increase from 84 at 25—35 to 148 at 55—65. The Class I rate is 5 % above average, and that for unskilled labour a little higher. Of the five graded classes III, skilled labour, returns the lowest ratio (88), but as in so many other cases the most favourable figures are those for miners and agricultural labourers. Exceptionally however in this case the figure for the former (61) is lower than for the latter (66). But, again as in many other cases, the miners' ratio shows a sudden increase after the apparently critical age for him of 55 is past. It should be noted also that at 25—35 the skilled workman, with a ratio of 68, seems to take as cheerful a view of life even as the agricultural labourer, the miner alone returning a better figure; but as age increases his suicide ratio increases with it, till at 55—65 it reaches 104.

If we inquire what manner of man it is who takes his own life we shall find that physical labour in general seems to act as a deterrent, the highest rates being returned by occupations not involving much output of physical energy. On the other hand, there is a distinct association with drink and drugs. The highest ratios for populations of over 10,000 men at all ages are as follows: barmen 226, photographers 221, chemists and druggists 211, publicans 200, barbers 200, general shopkeepers

184, doctors 174, costermongers 168, watchmakers 158, and lawyers 153. On the other hand, 15,442 tailors in Bethnal Green and Stepney, largely, of course, Jews, and 7318 tin miners did not furnish a single case of suicide, even though the latter have a higher death-rate from phthisis alone (ratio 482) than many occupations from all causes jointly. The rates for druggists and doctors seem to have been commented upon in advance by Shakespeare in what has become one of the most hackneyed of quotations, but that for lawyers suggests that other factors may influence the rate for doctors than the sight of means to do ill deeds. In contrast with the ratios for these professions that for the clergy is only 47, but certain important working-class occupations return even lower rates. Thus for motor drivers, a rather more numerous body of men than the clergy, the ratio is only 21.

Accident. The range of variation here is very wide, a few dangerous occupations producing a considerable proportion of the fatal accidents. The class ratios range from 48 % for Class I to 165 for Class V and 246 for Class VII (miners). The immunity of the middle class seems perhaps less unexpected than that of textile workers (49), with their constant exposure to risk from moving machinery. Yet of all the textile occupations distinguished, strippers and grinders (cotton) alone showed a rate in excess of average (ratio 115), and these were but a small body of 4011 men aged 25—65.

The contrast between skilled and unskilled labour is here very striking, the mortality from accident in Class V being exactly double that in Class III.

Apart from merchant sailors (415 %) and fishermen (202), whose rates are of doubtful value owing to want of correspondence between the population caught by the census and the deaths registered, the highest occupational ratios are as follows: bargemen 402, railway platelayers 265, coalminers 246, dockers 208, carmen, carriers 169, railway guards, porters, pointsmen and signalmen 163, and coalheavers 160, most of these, apart from miners, belonging to Class V. In most cases the risk is obvious, but the extent of the excess for bargemen and for carmen I confess surprises me. The mining excess is shared by iron and other miners (tin and lead) to an extent approaching that for coal.

Having now compared the mortalities from various causes recorded for different social strata, I shall very briefly refer to the proportion of total deaths in each case ascribed to various groups of causes (Table II). This method of dealing with the subject evades the difficulty created by overstatement of deaths as compared with population in Class V and their understatement in Class IV, though it is in other respects less satisfactory than the comparison of death *rates* hitherto relied on. However, the same facts are, generally speaking, brought out by both methods of statement, so that what follows will serve as a summarized recapitulation in another form of what has already been said.

Proportional statements of this kind have been prepared for each cause dealt with in the Report, but it must suffice now to use them for eight groups of causes—tuberculosis, cancer, nervous, circulatory, respiratory, digestive, and urinary diseases, and violence.

TABLE II.

Death-Rates from various Causes in Social Classes per Mille of those from all Causes at each Age distinguished.

(Mortality at 25—65 standardized.)

	Age	All Occupied and Retired Males	Social Class							
			I	II	III	IV	V	VI	VII	VIII
Tuberculosis	25—35	386	461	388	451	371	363	402	223	351
	35—45	279	290	253	327	270	282	275	192	263
	45—55	168	137	144	192	174	182	151	128	135
	55—65	78	58	63	89	79	95	76	61	57
	25—65	192	188	178	217	191	203	176	129	173
Cancer	25—35	24	27	21	26	24	22	20	18	28
	35—45	54	62	53	56	54	50	65	49	58
	45—55	116	133	108	120	125	106	107	103	131
	55—65	144	157	145	148	148	138	135	112	178
	25—65	99	112	95	104	103	91	100	82	115
Diseases of the Nervous System	25—35	61	59	61	57	61	68	50	47	72
	35—45	96	118	94	98	97	93	89	83	87
	45—55	114	134	120	115	110	107	129	93	100
	55—65	128	142	129	137	129	117	137	126	112
	25—65	107	122	108	1[illegible]	106	101	114	97	96
Diseases of the Circulatory System	25—35	78	73	81	71	81	87	93	66	65
	35—45	117	103	109	109	133	128	105	106	114
	45—55	152	146	149	150	153	154	161	158	174
	55—65	197	212	203	194	200	186	204	195	214
	25—65	150	151	149	147	154	148	160	148	156
Diseases of the Respiratory System	25—35	120	91	114	112	125	143	124	111	103
	35—45	139	105	135	130	136	164	142	140	137
	45—55	150	106	125	139	156	187	150	182	137
	55—65	174	114	142	165	181	213	174	233	150
	25—65	152	107	131	143	156	183	155	183	135
Diseases of the Digestive System	25—35	55	65	61	55	60	46	72	53	60
	35—45	60	83	84	56	55	48	68	45	56
	45—55	69	98	93	64	64	55	68	54	62
	55—65	62	85	78	57	59	51	57	53	62
	25—65	62	85	81	58	60	51	64	52	61
Diseases of the Urinary System	25—35	37	33	36	40	38	38	46	24	38
	35—45	49	51	53	55	50	42	58	43	46
	45—55	65	81	72	73	65	56	79	41	47
	55—65	72	89	84	78	67	61	78	54	56
	25—65	60	70	66	67	58	52	71	44	48
Suicide	25—35	30	43	40	23	34	27	29	20	31
	35—45	30	34	40	26	34	21	39	19	37
	45—55	24	33	34	24	21	18	28	15	28
	55—65	17	15	21	17	16	15	21	12	16
	25—65	24	29	32	21	25	20	27	15	25
Accident	25—35	103	49	78	70	97	111	51	310	107
	35—45	79	47	71	54	70	86	36	226	85
	45—55	55	31	50	40	48	66	24	141	72
	55—65	35	20	31	29	31	43	22	74	49
	25—65	61	33	52	42	54	70	29	158	72
Other Causes	25—35	106	99	120	95	109	95	113	128	145
	35—45	97	107	108	89	101	86	123	97	117
	45—55	87	101	105	83	84	69	103	85	114
	55—65	93	108	104	86	90	81	96	80	106
	25—65	93	103	108	89	93	81	104	92	119
Pulmonary Tuberculosis	25—35	356	428	358	420	340	336	367	196	312
	35—45	262	270	237	309	255	268	250	174	223
	45—55	157	126	135	181	163	172	139	118	121
	55—65	73	53	58	84	73	91	67	57	50
	25—65	179	174	165	204	178	192	160	117	151

TABLE II (*continued*).

	Age	All Occupied and Retired Males	Social Class							
			I	II	III	IV	V	VI	VII	VIII
Diabetes	25—35	12	17	13	15	11	9	13	11	11
	35—45	10	16	13	9	11	7	11	10	12
	45—55	11	15	17	10	9	6	10	7	8
	55—65	15	28	25	13	13	7	17	7	8
	25—65	13	20	19	12	11	7	14	8	9
Tabes Dorsalis and General Paralysis of the Insane	25—35	17	14	14	17	19	23	12	10	10
	35—45	39	51	37	39	41	39	27	32	15
	45—55	29	43	31	26	28	28	22	21	17
	55—65	12	18	13	12	14	11	9	9	10
	25—65	23	30	23	22	25	23	16	16	13
Cerebral Haemorrhage, Apoplexy	25—35	8	8	10	5	9	8	6	8	10
	35—45	21	21	23	23	19	19	28	17	16
	45—55	47	49	49	54	46	40	66	34	35
	55—65	73	78	73	80	75	64	84	75	58
	25—65	46	48	46	50	45	38	59	43	34
Valvular Disease of the Heart	25—35	32	28	33	30	31	36	39	32	21
	35—45	43	37	39	42	47	47	37	42	45
	45—55	51	45	45	52	51	55	51	54	62
	55—65	59	55	59	57	61	60	56	62	66
	25—65	49	45	47	49	50	52	49	51	53
Angina Pectoris and Arterio-Sclerosis	25—35	1	1	0	1	2	1	1	1	1
	35—45	3	3	4	3	3	4	3	3	2
	45—55	11	20	13	10	12	8	12	8	7
	55—65	22	41	25	22	21	15	24	17	14
	25—65	11	22	13	12	12	8	14	10	6
Aneurysm	25—35	4	3	4	3	8	4	2	1	6
	35—45	12	7	9	10	19	14	3	6	9
	45—55	11	9	12	11	14	13	5	10	5
	55—65	7	8	8	7	8	7	4	5	9
	25—65	9	7	8	8	12	10	4	5	9
Bronchitis	25—35	11	5	10	12	10	16	14	8	6
	35—45	23	11	18	21	25	32	29	25	7
	45—55	45	22	32	43	48	61	53	57	34
	55—65	82	36	59	79	87	107	90	121	55
	25—65	48	23	35	49	52	63	59	68	30
Pneumonia	25—35	97	73	92	88	103	114	100	92	80
	35—45	99	80	99	91	94	116	100	92	109
	45—55	85	66	73	76	86	108	75	96	83
	55—65	82	36	59	79	87	107	90	121	55
	25—65	85	68	78	75	84	102	79	88	83
Hernia	25—35	1	1	2	2	2	2	1	1	2
	35—45	2	1	3	2	1	2	1	2	6
	45—55	3	2	3	2	3	3	2	3	5
	55—65	4	3	4	4	4	4	3	6	5
	25—65	3	1	3	3	3	3	2	4	4
Cirrhosis of the Liver	25—35	4	3	7	4	4	4	4	3	2
	35—45	15	23	30	12	11	12	7	8	8
	45—55	23	36	44	19	20	16	18	13	10
	55—65	19	32	32	16	16	14	16	11	13
	25—65	16	27	31	14	14	12	14	10	9
Other Diseases of the Digestive System	25—35	50	61	52	49	54	40	67	49	56
	35—45	43	59	51	42	43	34	60	35	42
	45—55	43	60	46	43	41	36	48	38	47
	55—65	39	50	42	37	39	33	38	36	44
	25—65	43	56	47	41	42	35	49	39	47
Bright's Disease	25—35	23	23	24	25	24	24	33	14	23
	35—45	34	35	38	39	35	28	42	26	30
	45—55	48	62	55	56	47	40	63	26	32
	55—65	50	66	60	56	45	40	55	35	35
	25—65	42	52	48	49	40	35	52	28	32

Tuberculosis at 25—65 is almost entirely phthisis. The highest proportion of deaths in the standard population, 217 per thousand, is that of Class III, skilled workers, Classes V, IV, I, II, VI, VIII and VII following in the order named. The proportion for miners, 12·9 % as against an average of 19·2, is very low, but their deaths from violence are 17·3 %, the average being 8·5. Even if, however, we compensate for this by expressing their deaths from tuberculosis per cent. of deaths from *disease*, the proportion only comes to 15·3 %, that for all classes, similarly stated, being 20·5. At 25—35 Class I returns the highest proportion, 46·1 % of all deaths, and at 55—65 very nearly the lowest, 5·8. On the other hand Class V returns the lowest figure for any of the five graded classes at 25—35 and the highest for any class at 55—65, so the variation with social status of the age distribution of tubercle mortality is brought out again by this method of statement. The proportions for Class III are high at each age, and highest of all at 35—45 and 45—55.

Cancer mortality varies much less as between class and class than that from all causes, so the proportion of total mortality due to this cause tends to be low where total mortality is high (Class V) and high where it is low (Classes I and VIII). It is uniformly low however for miners, whose total mortality is below average at all ages up to 55, partly because of their high rates from accident, and partly because their cancer rates are less at each age than for any other class except agricultural labourers.

Nervous Diseases. On the whole the proportions returned are fairly uniform, but for each of the three later age groups Class I takes highest place, its proportion for the four jointly being 12·2 % as against a general average of 10·7, and a minimum of 9·6 for agricultural labourers. Next to Class I comes Class VI (11·4), its excess occurring entirely at the two later age groups.

Circulatory Diseases. Proportions vary only from a minimum of 14·8 (Class V) to a maximum of 16·0 (Class VI), the shares taken at different ages by this group of causes also varying little on the whole. But Class VIII, which returns the lowest proportion at 25—35, yields the highest at 45—55 and 55—65.

Respiratory Diseases. The association with poverty of this group of causes is well brought out by proportional statement. From a minimum of 10·7 % in Class I its share rises steadily to a maximum of 18·3 in Class V, amongst the five graded classes. That for miners is also 18·3, which implies that if allowance be made for their excess of deaths from violence it is highest of all. This excess for miners occurs entirely at the later ages, and especially at 55—65. The general social gradation applies not only at all ages jointly but at each age, the Class I share always being lowest and that of Class V highest, with little irregularity between.

Digestive Diseases. Here the position is exactly reversed for the five graded classes. To say that the poor man dies from his lungs and the rich man from his stomach would be a gross overstatement, but one which emphasizes a general tendency clearly impressed upon the figures. Amongst the five graded classes the highest proportion at every age is that of Class I and the lowest that of Class V, but the intermediate gradations are not so regular as for respiratory diseases. The

range, at all ages jointly, is from 8·5 % for Class I to 5·1 for Class V. The figures for miners are low at each age, but those for textile workers are high in the two earlier groups, being highest of all at 25—35.

Urinary Diseases (chiefly Bright's). The range of variation is neither regular nor great, but the proportion tends to rise as we ascend the social scale, from 5·2 % for Class V to 7·0 for Class I. Class VI returns the highest figure, 7·1, and Classes VII and VIII the lowest, 4·4 and 4·8. The tendency shared by all classes for the proportion to increase with advancing age is at a maximum in Class I and a minimum in Class V.

Violence. There is really in this case no social gradation to be made out. Generally speaking Classes I, III and VI return low ratios and V, VII and VIII high ones. The lowest of all is that for Class VI, those for Classes I and III, practically equal to each other, coming next at each age except 55—65, where Class I is lowest of all. The figures for miners are in great excess at each age, their very heavy ratio for accident swamping the effect of their light one for suicide, but this excess lessens greatly as age advances, partly because their relative position as regards disease deteriorates, and partly because their mortality from accident lessens rapidly with age—much more so than that of any other class. But even if neither of these two factors applied the ratio for accident would be bound to fall, as in all classes, as advancing age increases deaths from disease without increasing risk of accident.

MISCELLANEA.

(1) Note on Dr Burnside's Paper on Errors of Observation.

By ETHEL M. NEWBOLD.

In a recent paper on errors of observation* Dr Burnside suggests an alternative formula for the probable error of the experimental determination of astronomical or physical constants. His results are best given in his own words: "The standard formula for the probable error is arrived at by assuming (i) that the errors follow Gauss's law, (ii) that the precision-constant has that particular value which makes the probability of the observed set of values as great as possible. The alternative formula here put forward, which gives a smaller value for the probable error, is arrived at by making the same first assumption and replacing the second by the assumption that *à priori* all values of the precision-constant are equally likely. It is a matter of individual judgment which form of second assumption is the more reasonable."

Since the suggested formula deviates from the standard formula in the direction of less rather than more caution, it may be worth while to put forward a few considerations which reasonable individual judgment cannot leave out of account.

It may first be pointed out that the standard formula as quoted by Dr Burnside would not be applied by modern statisticians to samples of the size considered in his arithmetical examples, as the error in approximating to the standard deviation of the whole population of errors by that of the sample would not be without importance. The distribution of the means of small samples has been investigated by "Student"†, and he has furnished tables‡ for use in such cases. His results will be compared with that of Dr Burnside later. In such samples, therefore, Dr Burnside's formula, unless securely founded, will overestimate still more dangerously the stability of the experimental values.

Dr Burnside's first charge against the usual procedure is as follows:—"It would be difficult to justify the assumption that, because a particular value of the precision-constant makes the probability of the observed event as great as possible, the precision-constant necessarily has that value. Moreover a further examination will show that there is no need to make this particular assumption."

No one will deny the truth of both these statements, but they are beside the mark, and only demolish a straw dummy set up by the author himself. To assume that in our ignorance of the true value of certain constants, the most reasonable approximation is given by those values which make the probability of the observed results of carefully conducted experiments as great as possible, is by no means the same thing as to assume that the constants *necessarily have* those values.

As Dr Burnside himself admits, an assumption of some sort has to be made. To base this assumption on experience is in accordance with the everyday practice, either conscious or unconscious, of most reasonable people in all the common actions of life. Dr Burnside rejects this procedure in the case of the precision-constant h, but tacitly admits it in the case of the unknown value of the physical constant itself, by using for it the arithmetic mean of the observed values.

If we reject with Dr Burnside the assumption of maximum probability for the observed event, we have to consider (1) how far are we justified in the assumption of equal probability for every value of the precision-constant h; and (2) is this the only alternative? If not, how are we to choose between it and other apparently equally justifiable hypotheses?

* *Proc. Camb. Phil. Soc.*, 1923, Vol. xxi. Pt. V. pp. 482—487.

† "The Probable Error of a Mean" by "Student," *Biometrika*, Vol. vi. 1908, p. 1.

‡ *Loc. cit.* and *Biometrika*, Vol. xi. p. 414. See also Pearson's *Tables for Statisticians*, p. 36.

Consider (1) first. In the case of errors of observation, could one seriously suggest to a scientific observer in astronomy, or indeed in any other branch of physics, that he is in complete ignorance of the precision of his observations, and that this is equally likely to be anywhere between zero and perfect accuracy? Such a suggestion may seem to be casting a slur on his skill, but it is in reality *overestimating* instead of underestimating the accuracy of determination. The real value of h, being in most practical cases finite, however large it may be, is nearer to zero than to infinity, so that the assumption of a flat range of frequency from 0 to ∞ gives too many large values of h, i.e. on the whole too small a variation of x, hence Dr Burnside's result of a smaller probable error. A similar argument suggests that (as is shown later to be the case) an assumption of equal probability for all values of Σ (the standard deviation of the whole population of errors) would give too large a value for the probable error; for the real value of Σ, however large it is, is also nearer to zero than to infinity, so that a flat range from 0 to ∞ gives too many large values of Σ, which implies too large a variation of x.

The paper in question is confined to errors of observation, but even in variable qualities we have some sort of idea of the limits of variation expected in most of the usual subjects of statistical research. To give equal weight to all values seems to imply sampling from a much larger universe than is usually necessary in practical cases.

Suppose, however, that we waive this question also, and agree to assume that all values of the accuracy of determination are equally probable, is there no choice for the exact form of the parameter expressing variation? Dr Burnside writes the normal distribution in the form $\sqrt{\frac{h}{\pi}}e^{-hy^2}$ and makes the probability of all values of h constant. Is there not equal justification, for instance, for writing this in the form $\frac{1}{\Sigma\sqrt{2\pi}}e^{-\frac{y^2}{2\Sigma^2}}$ and assuming a constant probability for every value of Σ, instead of for every value of h?

Suppose we do this, and let $p'd\Sigma$ be the *à priori* probability that Σ lies between Σ and $\Sigma+\delta\Sigma$, then following the method of Dr Burnside's paper, and using his notation, we get the *à posteriori* probability that this is the case to be

$$kp'\left(\frac{1}{2\Sigma^2}\right)^{\frac{n-1}{2}}e^{-\frac{1}{2\Sigma^2}\left(s_2-\frac{1}{n}s_1^2\right)}d\Sigma;$$

and our new assumption gives

$$\frac{1}{kp'}=\frac{\Gamma\left(\frac{n-2}{2}\right)}{2^{\frac{n}{2}}\sigma^{\frac{n-2}{2}}}, \qquad \text{where } \sigma=s_2-\frac{1}{n}s_1^2.$$

Continuing on Dr Burnside's lines, the *à posteriori* probability that $\frac{1}{n}\overset{n}{S}_1(y_i)$ should lie between γ and $-\gamma$ is the integral of

$$\frac{2^{\frac{n}{2}}\sigma^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)}\frac{\left(\frac{1}{2\Sigma^2}\right)^{\frac{n-1}{2}}}{\pi^{\frac{n}{2}}}e^{-\frac{1}{2\Sigma^2}\left(\overset{n}{S}_1(y_i^2)+\sigma\right)}d\Sigma\,dy_1dy_2\ldots dy_n.$$

Integrating with respect to Σ and making the same transformations as in the original paper, the result is the integral of

$$\frac{\sigma^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)}\frac{1}{\pi^{\frac{n}{2}}}\frac{\Gamma(n-1)}{\left(\overset{n}{S}_1(z_i^2)+\sigma\right)^{n-1}}dz_1dz_2\ldots dz_n,$$

where the limits for z_1 are $\pm\gamma\sqrt{n}$ and for $z_2, z_3\ldots z_n$ are $\pm\infty$.

This gives for n even

$$\frac{1.3.5\ldots(n-5)(n-3)}{2.4.6\ldots(n-6)(n-4)}\int_0^{\gamma\sqrt{\frac{n}{\sigma}}}\frac{dt}{(1+t^2)^{\frac{n-1}{2}}},$$

and for n odd

$$\frac{2}{\pi}\frac{2.4.6\ldots(n-6)(n-4)}{1.3.5\ldots(n-5)(n-3)}\int_0^{\gamma\sqrt{\frac{n}{\sigma}}}\frac{dt}{(1+t^2)^{\frac{n-1}{2}}},$$

i.e. for any value of n,

$$\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\int_0^{\gamma\sqrt{\frac{n}{\sigma}}}\frac{dt}{(1+t^2)^{\frac{n-1}{2}}},$$

the final result being that the probability is $\frac{1}{2}$ when $\gamma=\theta_n\sqrt{\frac{\sigma}{n}}$ if θ_n is given by the equation

$$\int_0^{\theta_n}\frac{dt}{(1+t^2)^{\frac{n-1}{2}}}=\int_{\theta_n}^{\infty}\frac{dt}{(1+t^2)^{\frac{n-1}{2}}},$$

instead of, as in Dr Burnside's case, the equation

$$\int_0^{\theta_n}\frac{dt}{(1+t^2)^{\frac{n}{2}+1}}=\int_{\theta_n}^{\infty}\frac{dt}{(1+t^2)^{\frac{n}{2}+1}}.$$

Hence the new values for θ_n will be the same as those given in Mr David Gibb's table if each number in the n column is increased by 3, thus ·3249200 is the value of θ_n in samples of 4 when all values of h are considered equally likely, but in samples of 7 when all values of Σ are considered equally likely. The value of the probable error is now greater than that given by the standard form for the small values of n considered. This is in accord with expectation, for as was shown above this result is suggested by the assumptions made. It also follows that, as is otherwise obvious, h and Σ cannot both be equally likely to have all values. How are we to choose between these and other parameters? On our choice depends the value of the probable error.

Are we equally justified in passing from a discontinuous series to the conception of a continuous function of equal density in the case of either h or Σ? The mere size of the resulting probable error is no criterion of the reasonableness of our assumptions till it has been tested against practical experience. If we now generalise these results, and take S as our parameter, where $S^m=h$, and assume that all values of S are equally probable, and then give m different values, we see the relation in which Dr Burnside's formula stands to that found by "Student" in his paper in *Biometrika*, Vol. VI.

Following Dr Burnside's method as before, if $p''dS$ is the *à priori* probability that S lies between S and $S+dS$, then the *à posteriori* probability that this is the case is

$$kp''(S^m)^{\frac{n-1}{2}}e^{-S^m\left(s_2-\frac{1}{n}s_1^2\right)}dS,$$

where $\frac{1}{kp''}$ = the absolute value of $\frac{1}{m}\frac{\Gamma\left(\frac{n-1}{2}+\frac{1}{m}\right)}{\sigma\left(\frac{n-1}{2}+\frac{1}{m}\right)}$

(obtained by substituting x for σS^m and integrating from 0 to ∞). The *à posteriori* probability, that $\frac{1}{n}\sum_1^n y_i$ should lie between $-\gamma$ and γ, is the integral of

$$\frac{m\sigma^{\frac{n-1}{2}+\frac{1}{m}}}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n-1}{2}+\frac{1}{m}\right)}S^{m\left(n-\frac{1}{2}\right)}e^{-S^m\left(\sigma+\sum_1^n(y_i)^2\right)}dSdy_1dy_2\ldots dy_n.$$

Integrating with respect to S (with the help of the substitution $x = S^m(\sigma + \sum_1^n S(y_i^2))$) and making the same transformations as before, the result is the integral of constant $\times \dfrac{dz_1 dz_2 dz_3 \ldots dz_n}{\left(\sum_1^n S(z_i^2) + \sigma\right)^{n+\frac{1}{m}-\frac{1}{2}}}$

where the limits for z_1 are $\pm\gamma\sqrt{n}$ and for $z_2, z_3 \ldots z_n$ are $\pm\infty$.

This integral is reduced by putting $z_2 = \beta \tan\theta$ where $\beta^2 = \sigma + z_1^2 + z_3^2 + \ldots + z_n^2$ and continuing the process for each of the other z's except z_1; each step will reduce the index of the denominator by the term $-\frac{1}{2}$, hence the final result is constant $\times \int_0^{\gamma\sqrt{\frac{n}{\sigma}}} \dfrac{dt}{(1+t^2)^{n+\frac{1}{m}-\frac{1}{2}-\frac{n-1}{2}}}$,

$$\text{i.e. constant} \times \int_0^{\gamma\sqrt{\frac{n}{\sigma}}} \frac{dt}{(1+t^2)^{\frac{n}{2}+\frac{1}{m}}}.$$

The constant can be obtained either directly as before or by integrating from 0 to ∞, and equating the whole probability thus found to unity.

Dr Burnside's method generalised thus shows that the family of integral curves which give the probability P, that the deviation of the mean of the sample (measured in terms of the standard deviation of the sample) lies between $-\infty$ and z, is, on this hypothesis, given by

$$P = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{m}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}+\frac{1}{m}\right)} \int_{-\infty}^{z} \frac{dz}{(1+z^2)^{\frac{n}{2}+\frac{1}{m}}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or in the form

$$P = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{m}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}+\frac{1}{m}\right)} \int_{-\frac{\pi}{2}}^{\tan^{-1}z} \cos^{n+\frac{2}{m}-2}\theta\, d\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where n = number in the sample.

The formula found by "Student" for the same probability, by an entirely different method and *without* making any *à priori* assumption as to the distribution of the variation, was*

$$P = \frac{\Gamma\left(\frac{n}{2}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)} \int_{-\frac{\pi}{2}}^{\tan^{-1}z} \cos^{n-2}\theta\, d\theta,$$

and this can be deduced from (2) by putting $\frac{1}{m} = 0$. Hence we have the following schema:

TABLE I.

	(i)	(ii)	(iii)	(iv)	
Parameter which is assumed to have constant frequency for all values †	Σ^1 or $h^{-\frac{1}{2}}$	Σ^0 h^0	Σ^{-1} $h^{\frac{1}{2}}$	Σ^{-2} h^1	and so on
Value of $\frac{1}{m}$ in (1) or (2)	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	
		"Student's" case		Dr Burnside's case	

* *Biometrika*, Vol. VI. p. 11.

† The constants arising from the factor $\frac{1}{2}$ $\left(\text{since } h = \frac{1}{2\Sigma^2}\right)$, which appear in the intermediate steps, cancel out in the final results.

Thus though "Student's" formula falls neatly into its place in this series, it does so without burdening itself with the *à priori* assumptions attached to the other cases.

"Student" has himself tested his formula from experience, against two series of 750 means of random samples of 4 from approximately normal distributions, with satisfactory results. By means of the tables for this series of curves, which "Student" has provided, I have tested Dr Burnside's formula against the same data, and also the formulae obtained by putting $\frac{1}{m} = \pm\frac{1}{2}$. These results are given in Table II.

Practical Test of Different Hypotheses.

TABLE II.

Distribution of Means (in terms of S.D. *of sample) of Random Samples of* 4 *from Height and Left Middle Finger Measurements of* 3000 *Criminals.*

(Data of W. R. Macdonell, *Biometrika*, Vol. I. p. 219, and used by "Student" in his paper in *Biometrika*, Vol. VI. p. 1.)

	Parameter for which a straight line frequency distribution is assumed	Form of the Integral Curve. Area up to z
(i)	Σ^1 or $h^{-\frac{1}{2}}$	$\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}-1\right)}\int_{-\frac{\pi}{2}}^{\tan^{-1}z}\cos^{n-3}\theta\,d\theta$
(ii)	Σ^0 or h^0	$\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\int_{-\frac{\pi}{2}}^{\tan^{-1}z}\cos^{n-2}\theta\,d\theta$ = "Student's" formula
(iii)	Σ^{-1} or $h^{\frac{1}{2}}$	$\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\int_{-\frac{\pi}{2}}^{\tan^{-1}z}\cos^{n-1}\theta\,d\theta$
(iv)	Σ^{-2} or h^1	$\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\int_{-\frac{\pi}{2}}^{\tan^{-1}z}\cos^{n}\theta\,d\theta$ = Dr Burnside's formula

Example I. *Height.*

z	Observed	(i)	(ii)	(iii)	(iv)
less than $-3{\cdot}05$	9	18·7	5	1·4	·4
$-3{\cdot}05$ to $-2{\cdot}05$	14·5	19·4	9·5	4·3	1·8
$-2{\cdot}05$ to $-1{\cdot}55$	11·5	22·0	13·5	8·0	4·5
$-1{\cdot}55$ to $-1{\cdot}05$	33	43·7	34·5	25·5	18·1
$-1{\cdot}05$ to $-\ {\cdot}75$	43·5	46·7	44·5	39·4	33·6
$-\ {\cdot}75$ to $-\ {\cdot}45$	70·5	71·1	78·5	79·4	77·7
$-\ {\cdot}45$ to $-\ {\cdot}15$	119·5	98·0	119	134·6	146·0
$-\ {\cdot}15$ to $+\ {\cdot}15$	151·5	110·9	141	164·9	185·6
$+\ {\cdot}15$ to $+\ {\cdot}45$	122·5	98·0	119	134·6	146·0
$+\ {\cdot}45$ to $+\ {\cdot}75$	67·5	71·1	78·5	79·4	77·7
$+\ {\cdot}75$ to $+1{\cdot}05$	49·5	46·7	44·5	39·4	33·6
$+1{\cdot}05$ to $+1{\cdot}55$	26·5	43·7	34·5	25·5	18·1
$+1{\cdot}55$ to $+2{\cdot}05$	16·5	22·0	13·5	8·0	4·5
$+2{\cdot}05$ to $+3{\cdot}05$	10	19·4	9·5	4·3	1·8
more than $3{\cdot}05$	6	18·7	5	1·4	·4
$\chi^2=$	—	61·5	12·4	109·4	301·0
$P=$		$<{\cdot}000001$	·56	$<{\cdot}0000001$	$<{\cdot}0000001$

Example II. *Finger Measurements.*

z	Observed	(i)	(ii)	(iii)	(iv)
less than $-3{\cdot}05$	4	18·7	5	1·4	·4
$-3{\cdot}05$ to $-2{\cdot}05$	15·5	19·4	9·5	4·3	1·8
$-2{\cdot}05$ to $-1{\cdot}55$	18	22·0	13·5	8·0	4·5
$-1{\cdot}55$ to $-1{\cdot}05$	33·5	43·7	34·5	25·5	18·1
$-1{\cdot}05$ to $-\ {\cdot}75$	44	46·7	44·5	39·4	33·6
$-\ {\cdot}75$ to $-\ {\cdot}45$	75	71·1	78·5	79·4	77·7
$-\ {\cdot}45$ to $-\ {\cdot}15$	122	98·0	119	134·6	146·0
$-\ {\cdot}15$ to $+\ {\cdot}15$	138	110·9	141	164·9	185·6
$+\ {\cdot}15$ to $+\ {\cdot}45$	120·5	98·0	119	134·6	146·0
$+\ {\cdot}45$ to $+\ {\cdot}75$	71	71·1	78·5	79·4	77·7
$+\ {\cdot}75$ to $+1{\cdot}05$	46·5	46·7	44·5	39·4	33·6
$+1{\cdot}05$ to $+1{\cdot}55$	36·0	43·7	34·5	25·5	18·1
$+1{\cdot}55$ to $+2{\cdot}05$	11	22·0	13·5	8·0	4·5
$+2{\cdot}05$ to $+3{\cdot}05$	9	19·4	9·5	4·3	1·8
more than $3{\cdot}05$	6	18·7	5	1·4	·4
$\chi^2=$	—	54·54	7·39	84·6	347·1
$P=$		·000004	·92	$<$·000001	$<$·0000001

The values of χ^2 and P, obtained from Professor Pearson's Goodness of Fit Test, show clearly that "Student's" formula is in both cases the only one which corresponds with these observations. Dr Burnside's assumption gives the worst fit of the four cases tried, but it is clear that as $\frac{1}{m}$ increases in absolute value, the deviations of the theoretical from the observed values will become greater. This result is also in accordance with our expectation indicated above, that the choice of any positive power of h for the parameter will give too small a value for the probable error, and that of any negative power of h, i.e. a positive power of Σ, too large a value.

The sum of the above arguments—backed by practical experience—is that very little scope is left for individual judgment in choosing between "Student's" formula and that of Dr Burnside.

I should like to express my thanks to Dr Major Greenwood who has verified my algebra, and at whose suggestion this note was written.

(ii) On the Inheritance of Cranial Anomalies.

By PERCY STOCKS, M.D.

PART I.

In craniological memoirs it has been customary to provide among the published data of individual crania some account of the anomalies observed. To some readers these accounts seem of little importance, and this view is not without justification so long as we are uncertain whether slight cranial variations are inherited. I have therefore had long in view the desirability of studying, where feasible on the living head, the heredity of such anomalies. It seems very improbable, except in the rarest isolated cases, that we can obtain the skulls of persons known to be related. If, however, anomalies of the skull can be shown to be hereditary characters, the advantage of tabling cranial anomalies becomes obvious. Such variations, however we suppose them originally acquired, are transmitted and these cranial anomalies therefore become of great importance for the problem of evolution. A great deal may be learnt by carefully feeling over the head of a subject, and by pursuing the matter further, if any anomaly is noted, among the relatives of the subject. It must always be remembered that to the discriminating touch much

may be found which totally fails to make its presence appreciated to eye or to tape or to callipers.

The two cases here given are rather indicative of what is required in the collections that we desire, than very important in themselves. Post-coronal constriction and post-coronal depression, very common in English skulls (and at one time attributed to wearing a tight band round the head in childhood), ought to be easily noted and might well be followed up.

I have to thank my colleague Dr Percy Stocks for the following first contributions to the series I should like to see inaugurated.

KARL PEARSON.

I. *A Case of Depression of the Obelion with Hereditary History.*

Details are here given of a case recently seen at the Anthropometric Laboratory, which presented a well-marked depression of the cranial vault in the region of the obelion; this anomaly appeared to be definitely inherited according to information furnished by the subject himself.

The subject (IV. 14 in pedigree) was a male student of University College, aged 17¾; his father was of Polish nationality, his mother a Russian, of whom he was the only child. His height was 176·7 cm., weight 67·6 kgm., breadth of shoulders 39·9 cm., vital capacity 4030 c.c.

In a series of mental tests measuring speed of association, memory and visual judgment, he was somewhat below average, in accuracy and speed of counting above average, in all tests of sensory discrimination and muscular control very poor, but in muscular strength above average for male students. His visual acuity was normal though refraction showed slight hypermetropic astigmatism. Auditory acuity average. Circulatory system apparently normal. His head measurements were as follows:

Maximum length	191·2 mm.
„ breadth	156·8 „
Vertical auricular height	137·3 „
Horizontal circumference	572·0 „
Sagittal „	339·0 „
Auricular passages to nasion	95·5 „
„ „ cornea	90·0 „
„ „ supraciliary ridge	103·7 „
Internal ocular distance	34·6 „
External „ „	117·1 „
Interpupillary „	63·0 „

Whilst measuring his head, Professor Karl Pearson remarked the presence of a clearly defined depression in the region of the obelion, and on questioning him later, a history of similar peculiarity in other members of his family was elicited, which has been embodied in the accompanying pedigree and the details given below. A contour of the head in the sagittal plane was taken by means of a lead tape and is reproduced in Fig. 1.

The family history obtained from the subject was as follows: Of I. 1—4, his paternal great-grandparents, nothing was known, nor of a great-uncle II. 2. II. 3 had, according to one of his sons, almost certainly a depression at the back of his head, similar to that of IV. 14. II. 4—5 were normal, as were also III. 2, 5, 6, 9, 11—19. III. 1 was said to have been afflicted with some form of insanity and to have committed suicide. III. 3, 7, two sisters of his father, were believed by IV. 14 to present a similar anomaly. III. 4, 8, 10, all had a similar depression of the obelion; the first two were unaware of any abnormality in themselves, but had noticed it in III. 10, and had attributed it to an accident in childhood; III. 10 was suffering from melancholia with delusions of prosecution. IV. 1, 2, 5—8, 12, 13, first-cousins of IV. 14, were not certainly known to present the anomaly, though IV. 14 suspected it occurred in some of them. IV. 4, a married cousin, with her three children V. 1—3, and IV. 9—11 (three sons of III. 8), all presented the same depression. IV. 15, 16 were normal.

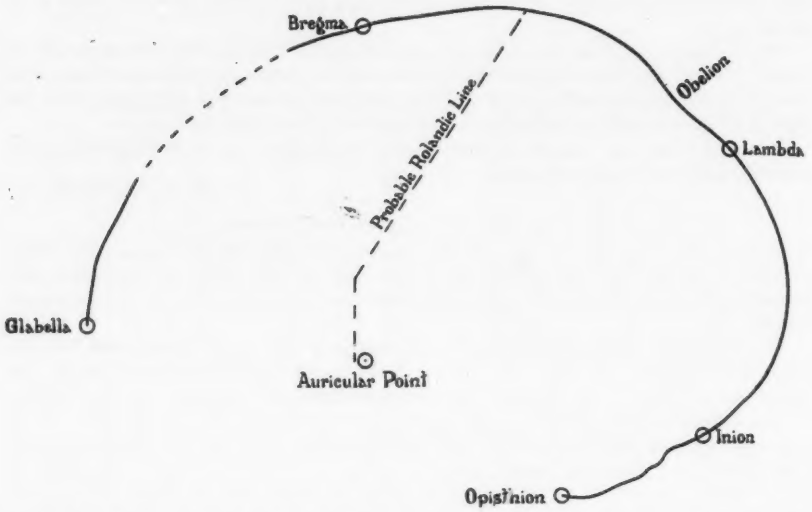


Fig. 1. Moorfields L. S. 118. English skull to show the depression of the Obelion. Reduction about $\frac{1}{12}$ths linear.

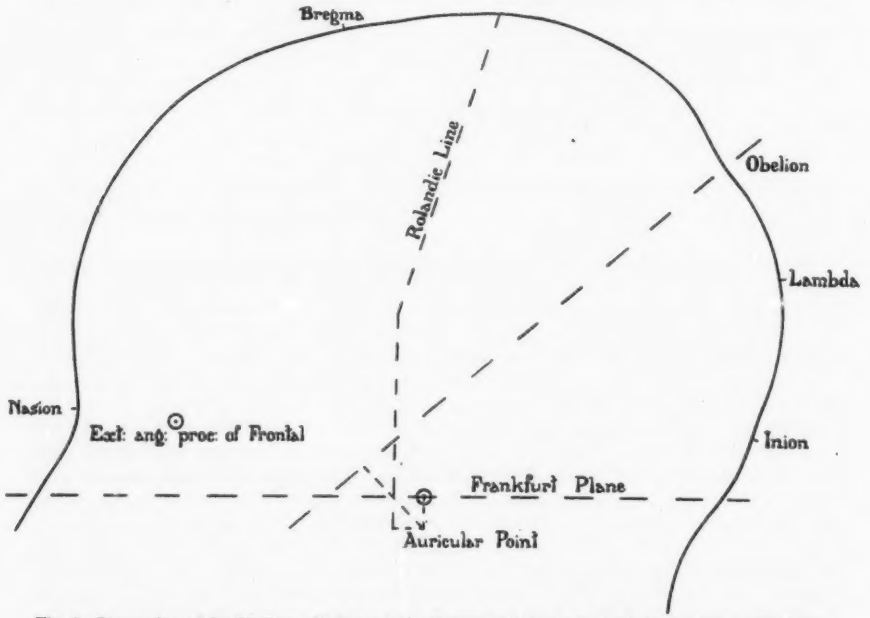
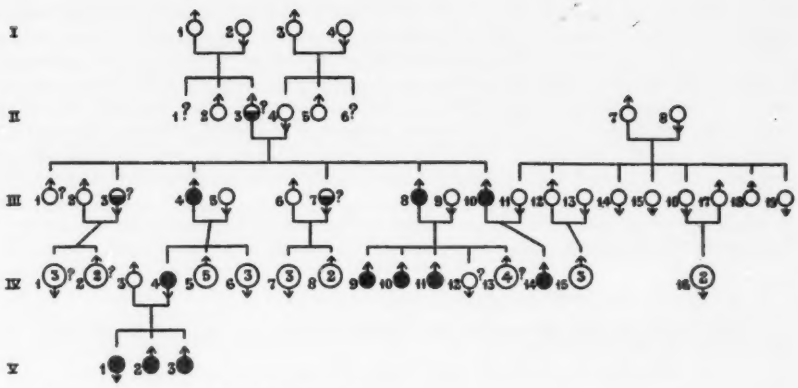


Fig. 2. Depression of the Obelion. Living subject, Russo-Polish origin. Reduction about $\frac{1}{12}$ths linear.

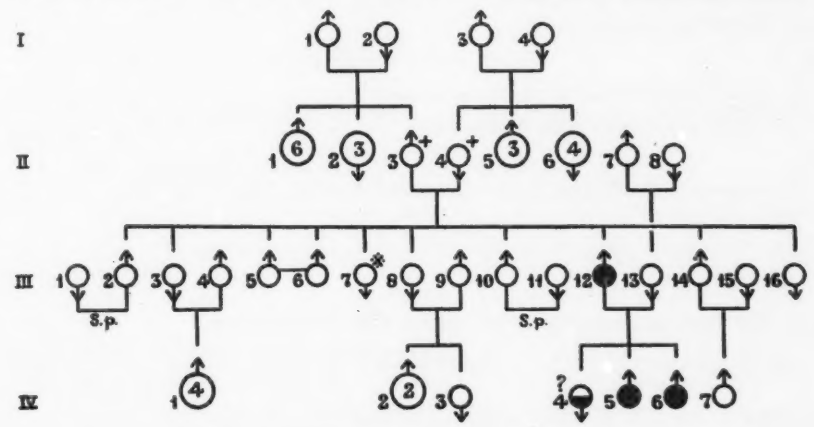


II 2, 3, & IV 4-6 order of birth unknown.

Pedigree No. 1.

II. *A Case of Occipital Prominence.*

Another case of hereditary cranial anomaly of a slighter character recently seen at the Anthropometric Laboratory (through the courtesy of Dr E. S. Barton) is depicted in Pedigree No. 2, and such scanty details as it was possible to obtain of the family are here given. III. 12, R. L., presented a distinctly palpable occipital prominence below the lambda, with a corresponding depression in the region of the latter. His head measurements were length 193 mm., breadth 144, height 139, I_1*=74·5. He denied knowledge of any similar anomaly in his parents or any



+ Position in family not known.
* Died at 2 years.

Pedigree No. 2.

* Cephalic Index.

of his brothers and sisters or their children. He also denied any history of epilepsy, mental defect or insanity in himself or any of his family. III. 13, his wife, appeared to have a normal head. IV. 4, E. L., aged 5 years, had according to her mother an obvious occipital prominence in infancy, but this had become less evident as she grew up; it could not be detected on examination. There was however a striking difference between the shape of her head and that of her father and two brothers; it was distinctly brachycephalic (length 164 mm., breadth 139, height 120, $I_1 = 84{\cdot}7$) and actually shorter antero-posteriorly than the head of her 2-year old brother. IV. 5, R. L., aged 2 years, presented a well-marked occipital prominence similar to his father (length of head 168 mm., breadth 129, $I_1 = 76{\cdot}8$). IV. 6, H. L., aged 6 months, also had an unusually prominent occiput (with a head length approximately 146 mm., breadth 110, giving an index about 75). He had adenoids. There were no signs of rickets in IV. 4—6.

(iii) Note on the Moments of a Binomial $(p+q)^n$ about its Mean.

By Prof. V. ROMANOVSKY (of the University of Turkestan).

In an editorial article in *Biometrika*, Vol. XII. footnote, p. 270, we read the following statement: "A simple reduction formula for the moments of a binomial $(p+q)^n$ about its mean was sought in vain. After a good deal of energy had been spent, we believe that μ_s being the sth moment about the mean

$$\mu_s = \left[\frac{d^s}{dx^s}\left(qe^{px}+pe^{-qx}\right)^n\right]_{x=0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is, perhaps, the easiest expression for reaching these moment-coefficients by successive differentiation." In 1914, being occupied with some questions in the theory of probability, I found a reduction formula for reaching the moments μ_s, which is much easier for calculations than the expression (1). With some other results it was communicated in 1915 to the Society of Naturalists of the University of Warsaw.

This reduction formula is

$$\mu_{s+1} = pq\left[\frac{d\mu_s}{dq} + ns\mu_{s-1}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

It can be demonstrated in the following manner.

According to (1) we see that the function

$$U^n = (qe^{px}+pe^{-qx})^n$$

is the generating function for the moments μ_s, so that we can write

$$U^n = \mu_0 + \frac{x}{1}\mu_1 + \frac{x^2}{2!}\mu_2 + \ldots$$

$$= \sum_{s=0}^{\infty} \frac{x^s}{s!}\mu_s.$$

Now differentiate both sides of this equation with regard to q, then

$$nU^{n-1}\frac{dU}{dq} = \Sigma \frac{x^s}{s!}\frac{d\mu_s}{dq} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

But

$$\frac{dU}{dq} = \frac{d}{dq}\left(qe^{px}+pe^{-qx}\right)$$

$$= e^{px} - e^{-qx} - x\left(qe^{px}+pe^{-qx}\right)$$

$$= \frac{1}{pq}\frac{dU}{dx} - xU,$$

for

$$\frac{dU}{dx} = pq\left(e^{px}-e^{-qx}\right).$$

Replace in (3) the $\frac{dU}{dq}$ by its value and we find

$$\frac{1}{pq} \cdot nU^{n-1}\frac{dU}{dx} - nxU^n = \Sigma \frac{x^s}{s!}\frac{d\mu_s}{dq},$$

or

$$\frac{1}{pq}\frac{d}{dx}(U^n) - nxU^n = \Sigma \frac{x^s}{s!}\frac{d\mu_s}{dq}.$$

Substituting here U^n by the series which it represents and performing the differentiation indicated in the first part of this equation we reach the new one:

$$\frac{1}{pq}\Sigma \frac{x^{s-1}}{(s-1)!}\mu_s - n\Sigma \frac{x^{s+1}}{s!}\mu_s = \Sigma \frac{x^s}{s!}\frac{d\mu_s}{dq},$$

which must hold identically. Therefore the coefficients of x^s on both sides must be equal for all integer positive values of s. We find accordingly that:

$$\frac{1}{pq}\frac{\mu_{s+1}}{s!} - \frac{n\mu_{s-1}}{(s-1)!} = \frac{1}{s!}\frac{d\mu_s}{dq}.$$

Multiplying here all members with $pq \cdot s!$ and rearranging we obtain the reduction formula

$$\mu_{s+1} = pq\left(\frac{d\mu_s}{dq} + ns\mu_{s-1}\right) \quad \ldots\ldots\ldots\

and generally

$$\omega^{(g)}_{h,k}=h\omega^{(g)}_{h,k-1}+(h-1)\,\omega^{(g-1)}_{h-1,k-1}+(k-1)\,(\omega^{(g)}_{h-1,k-2}+\omega^{(g-1)}_{h-2,k-2}) \quad (g+1\leqq h\leqq k)\ldots(10).$$

III. We find that :

$$N_{r,2r}=1\,.\,3\,.\,5\ldots(2r-1),$$

and therefore, by the first of equations (5),

$$\lim_{n=\infty}\frac{\mu_{2r}}{(\sqrt{2npq})^{2r}}=\frac{1\,.\,3\,.\,5\ldots(2r-1)}{2^r}$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}x^{2r}e^{-x^2}\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The second of equations (5) gives us

$$\lim_{n=\infty}\frac{\mu_{2r+1}}{(\sqrt{2npq})^{2r+1}}=0$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}x^{2r+1}e^{-x^2}\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

We see by (11) and (12) that the quantity

$$\frac{m-np}{\sqrt{2npq}},$$

where m represents the frequency of an event in n trials having the probability in each trial equal to p, is such that the mathematical expectation of any positive integer power of

$$\left(\frac{m-np}{\sqrt{2npq}}\right)^s$$

has the limit

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}x^s e^{-x^2}\,dx,$$

for n increasing indefinitely. Thus, remembering the well-known theorem of Tchebycheff-Markoff, we conclude that the probability of the inequalities

$$t_1\leqq\frac{m-np}{\sqrt{2npq}}\leqq t_2,$$

for any given t_1 and t_2, tends to

$$\frac{1}{\sqrt{\pi}}\int_{t_1}^{t_2}e^{-x^2}\,dx,$$

for $\lim n=\infty$, which is the theorem of Laplace.

I conclude with these remarks, omitting other applications which can be made with the reduction formula for the μ_s.

BERLIN, 1 *Nov.* 1923.

(iv) On the Time-interval between the Births of Children.

BY E. C. RHODES.

(i) *The Changes with Time.*

1. The material which forms the basis of the discussion in this paper was taken from the records of the Whitney Family*. These records giving the dates of births, marriages and deaths of the various members of this family and the corresponding data respecting their children, extend from the seventeenth century until the end of the nineteenth century. For the purposes of the writer each family was considered as a unit, and the information respecting the ages of husband and wife at marriage, the date of the marriage, and the successive intervals between the births of their children were extracted. Cases were rejected where the age of the wife at marriage was 31 years or more, and only those cases were taken in which the husband and wife had survived

* *Whitney, The Descendants of John Whitney who came from London, England to Watertown, Massachusetts in* 1635. By F. C. Pierce. 1895.

after 14 years 3 months of married life. In this way we have many cases showing the number of children born in a definite period (14 years 3 months), and the time elapsing between the births of successive children.

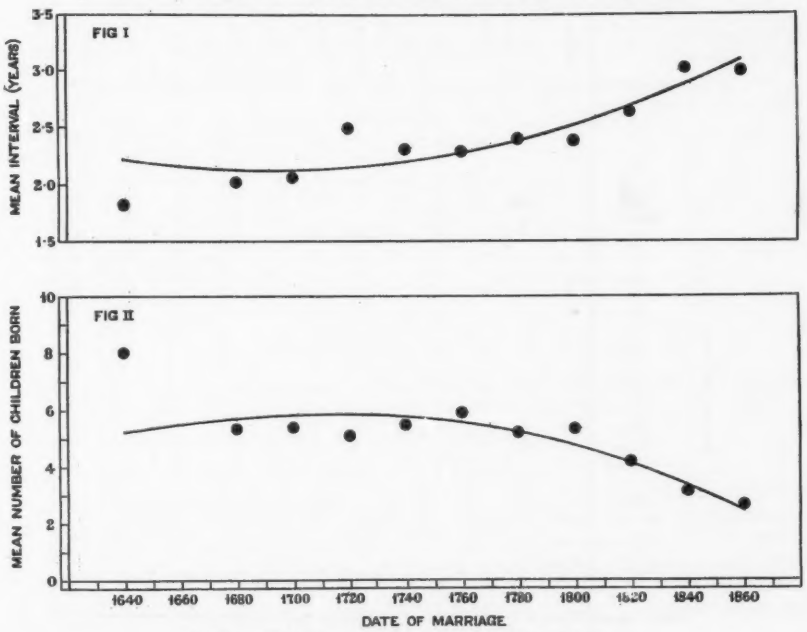
2. In the present section the writer deals with the alteration of the average time-interval between births with the date of marriage. The cases are grouped according to the date of marriage in 20-year intervals, and in each group the cases are further examined to get the distribution of the interval between births, these being grouped in $\frac{1}{2}$ year classes. Table I shows the distribution obtained. A word of explanation is necessary concerning the variable-intervals. Along the horizontal, all marriages which took place in the years 1800, 1801, ... 1819, are placed together in the group denoted by 1800—. Along the vertical all intervals of length from 1 year 0 days to 1 year 182 days inclusive are grouped together in the 1— year group. And so on with the other groups. The mean interval for each group is shown below.

Mean Interval between Births.

Date of Marriage.

	1640—	*1660—*	*1680—*	*1700—*	*1720—*	*1740—*	*1760—*	*1780—*	*1800—*	*1820—*	*1840—*	*1860—*
Mean interval (years)	1·82	—	2·02	2·06	2·48	2·31	2·29	2·40	2·39	2·64	3·02	2·99

MEAN INTERVAL BETWEEN CONFINEMENTS &
MEAN NUMBER OF CHILDREN BORN WITH DATE OF MARRIAGE



1640 signifies all years 1640—1659 inclusive and so on.

It will be seen that the mean interval between births on the whole increases with the date of marriage. If these means are shown diagrammatically this fact is more easily brought out. This is done in Fig. 1, and in this figure the points are shown with the best fitting cubic (weights being attached to individual means according to the corresponding number of occurrences). The complementary fact that the average number of children, born in a fixed length of time from the date of marriage, decreases over the period is shown in Table II. The means obtained from this table are shown diagrammatically in Fig. 2, and it will be seen that there is a general tendency for the mean number of children born to decrease.

Mean Number of Children to each Marriage.

Date of Marriage.

	1640—	*1680*—	*1700*—	*1720*—	*1740*—	*1760*—	*1780*—	*1800*—	*1820*—	*1840*—	*1860*—
Mean number of Births	8·00	5·33	5·38	5·08	5·46	5·88	5·19	5·30	4·14	3·12	2·63

Both figures show that this alteration in the interval between births may be said to date from the middle of the eighteenth century, since which time there has been an increase in the average interval between births in the data under discussion. This may be due to some unknown effect of environment on the fertility of these families or may be due to close inter-marriage amongst these Quaker Families.

TABLE I.

Distribution of Intervals and Date of Marriage.

Date of Marriage.

Length of Interval in years.	*1640*—	*1660*—	*1680*—	*1700*—	*1720*—	*1740*—	*1760*—	*1780*—	*1800*—	*1820*—	*1840*—	*1860*—
Less than *1*	—	—	—	1	1	2	3	2	8	8	9	6
1 —	2	—	2	8	4	17	18	5	53	50	69	91
1½ —	2	—	4	9	8	23	47	43	141	162	181	166
2 —	3	—	5	11	14	34	53	56	145	184	204	145
2½ —	—	—	2	3	13	7	19	24	56	91	130	92
3 —	—	—	—	2	1	9	6	11	32	60	88	56
3½ —	—	—	—	1	2	4	4	5	12	22	52	37
4 —	—	—	—	1	2	4	3	7	16	23	42	37
4½ —	—	—	—	—	1	—	2	3	9	15	23	43
5 —	—	—	—	—	1	1	—	—	5	13	23	22
5½ —	—	—	—	—	—	1	—	—	1	16	20	14
6 —	—	—	—	—	—	1	1	—	2	6	14	12
6½ —	—	—	—	—	—	—	—	—	3	1	12	10
7 —	—	—	—	—	—	—	1	—	2	2	9	10
7½ —	—	—	—	—	—	—	—	—	—	4	6	6
8 —	—	—	—	—	—	—	—	—	1	1	7	6
8½ —	—	—	—	—	—	—	—	—	1	—	6	5
9 —	—	—	—	—	—	—	—	—	—	—	4	3
9½ —	—	—	—	—	—	—	—	—	—	1	—	1
10 —	—	—	—	—	—	—	—	—	1	1	2	—
10½ —	—	—	—	—	—	—	—	—	—	—	4	—
11 —	—	—	—	—	—	—	1	—	—	—	4	—
11½ —	—	—	—	—	—	—	—	—	—	—	2	—
12 —	—	—	—	—	—	—	—	—	—	—	—	1
12½ —	—	—	—	—	—	—	—	—	—	—	—	—
13 —	—	—	—	—	—	—	—	—	—	—	—	1
Totals	7	—	13	36	47	103	158	156	488	660	911	764

3. Numerical Results. Table I. Correlation Coefficient $r = \cdot 168$

„ Ratio $\eta = \cdot 200$

Probable errors to these figures $\pm \cdot 013$.

Ordinates to Cubic, Fig. 1:

2·22 2·16 2·13 2·13 2·15 2·21 2·29 2·40 2·53 2·70 2·88 3·10.

Actual Means:

1·82 — 2·02 2·06 2·48 2·31 2·29 2·40 2·39 2·64 3·02 2·99.

TABLE II.

Frequency of Occurrence of Births and Date of Marriage.

Date of Marriage.

No. of Births.	*1640–*	*1660–*	*1680–*	*1700–*	*1720–*	*1740–*	*1760–*	*1780–*	*1800–*	*1820–*	*1840–*	*1860–*
0	—	—	—	1	—	2	—	1	3	9	42	58
1	—	—	—	—	—	—	1	1	1	14	37	74
2	—	—	1	—	2	—	—	1	5	22	87	90
3	—	—	—	—	—	2	3	2	10	28	92	90
4	—	—	—	1	1	5	2	5	15	43	52	60
5	—	—	—	1	2	4	5	7	23	38	55	43
6	—	—	1	2	6	3	7	13	26	31	41	14
7	—	—	—	2	1	5	12	6	22	21	9	11
8	1	—	1	1	—	2	2	1	7	4	3	2
9	—	—	—	—	—	—	1	—	3	—	—	—
10	—	—	—	—	—	1	—	—	—	—	—	1
Totals	1	—	3	8	12	24	33	37	115	210	418	443
Mean No. of Births	8·00	—	5·33	5·38	5·08	5·46	5·88	5·19	5·30	4·14	3·12	2·63

Table II. Correlation Coefficient $r = \cdot 424$

„ Ratio $\eta = \cdot 467$

Probable errors to these figures $\pm \cdot 015$.

Ordinates to Cubic, Fig. 2:

5·23 5·51 5·66 5·83 5·84 5·75 5·55 5·22 4·76 4·15 3·40 2·48.

Actual Means:

8·00 — 5·33 5·38 5·08 5·46 5·88 5·19 5·30 4·14 3·12 2·63.

It is seen that the difference between r and η is hardly significant. This is explained by the fact that the straight lines fitting these two series of points do not differ much from the cubics from 1740 onwards, in which period we get the most weighty figures of the series.

(ii) *The Changes according to the Number of Births and the Order of the Interval.*

1. The material which forms the basis of the discussion in the present section is the Whitney data already referred to. In the previous section it was found that the average interval between confinements increased over the period considered, the rate of increase being slow. In the present section the writer has found that the average interval differs according as it lies between first and second births, second and third and so on. The material for the purposes of the present inquiry was grouped first according to the number of births occurring in the fixed period, and

then according to the order of intervals. Table III shows the distributions which result from these groupings. When there are two births in the period, there is only one interval; when there are three births, there are two intervals, that between the first and second birth and that between the second and third; and so on.

TABLE III.

Distribution of Intervals between Births, according to the Number and Order of Intervals in a period of 14¼ years.

Number of Births	2	3		4			5				6					7					
Length of Interval in years. Interval ...	1st	1st	2nd	1st	2nd	3rd	1st	2nd	3rd	4th	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th	6th
Less than 1	2	2	—	2	1	—	1	1	2	1	2	3	2	—	—	1	2	3	1	2	2
1 —	10	17	15	15	11	7	26	16	10	4	24	13	6	10	7	28	14	9	10	4	4
1½ —	21	43	17	37	40	25	54	41	38	27	53	38	33	33	18	33	31	28	27	31	19
2 —	30	37	26	40	33	35	45	46	48	38	40	51	59	49	36	23	34	41	30	36	40
2½ —	22	30	21	32	30	21	19	23	35	19	12	24	21	32	39	3	2	7	11	10	16
3 —	12	22	24	21	19	22	13	22	15	24	7	7	13	9	16	—	4	—	9	2	3
3½ —	8	11	10	3	11	14	10	15	10	12	3	2	7	2	14	—	1	1	—	3	1
4 —	17	9	15	13	16	15	3	6	5	20	—	1	—	5	7	—	—	—	1	—	2
4½ —	15	10	15	8	4	11	4	5	5	11	—	—	1	2	5	—	—	—	—	—	—
5 —	7	10	13	4	5	8	—	—	4	10	—	1	—	1	1	—	—	—	—	—	1
5½ —	12	6	15	5	2	5	1	1	1	4	—	—	—	—	—	—	—	—	—	—	—
6 —	4	6	14	—	3	4	—	—	2	3	—	—	—	—	—	—	—	—	—	—	—
6½ —	6	7	7	—	2	3	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—
7 —	9	5	4	—	2	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7½ —	6	1	6	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8 —	7	4	3	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8½ —	4	2	3	—	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9 —	1	—	6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9½ —	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
10 —	1	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
10½ —	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
11 —	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
11½ —	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
12 —	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
12½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
13 —	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	205	223	220	180	180	180	176	176	175	174	141	140	142	143	143	88	88	89	89	88	88
Mean Interval (years)	4·45	3·32	4·31	2·70	2·91	3·47	2·27	2·51	2·60	3·17	2·00	2·18	2·33	2·39	2·75	1·75	1·96	2·01	2·16	2·13	2·29

TABLE III—(*continued*).

Number of Births	8							9								10								
Interval ...	1st	2nd	3rd	4th	5th	6th	7th	1st	2nd	3rd	4th	5th	6th	7th	8th	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
Less than 1	1	1	—	1	—	1	2	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
1 —	15	4	8	2	4	3	5	2	1	1	—	—	—	—	—	1	1	2	2	2	2	2	1	1
1½ —	5	15	10	14	12	10	9	—	2	3	4	4	3	3	3	—	1	—	—	—	—	—	—	—
2 —	1	2	4	5	6	8	7	—	—	—	—	—	1	1	1	1	—	—	—	—	—	—	—	—
2½ —	1	1	1	1	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—
3 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
3½ —	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
5 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
5½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
10 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
10½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
11 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
11½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
12 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
12½ —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
13 —	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	23	23	23	23	24	24	24	4	4	4	4	4	4	4	4	2	2	2	2	2	2	2	2	2
Mean Interval (years)	1·46	1·72	1·71	1·83	1·88	1·91	1·81	1·13	1·44	1·62	1·75	1·75	1·87	1·88	1·88	1·75	1·50	1·25	1·25	1·25	1·25	1·25	2·00	1·13

Length of Interval in years.

2. The average intervals are shown in Table III but are reproduced here in a form where comparisons are easier.

TABLE IV.

Mean Interval (in years).

No. of Births.	Interval. 1st	2nd	3rd	4th	5th	6th	7th	8th	9th
2	4·45	—	—	—	—	—	—	—	—
3	3·32	4·31	—	—	—	—	—	—	—
4	2·70	2·91	3·47	—	—	—	—	—	—
5	2·27	2·51	2·60	3·17	—	—	—	—	—
6	2·00	2·18	2·33	2·39	2·75	—	—	—	—
7	1·75	1·96	2·01	2·16	2·13	2·29	—	—	—
8	1·46	1·72	1·71	1·83	1·88	1·91	1·81	—	—
9	1·13	1·44	1·62	1·75	1·75	1·87	1·88	1·88	—
10	1·75	1·50	1·25	1·25	1·25	1·25	1·25	2·00	1·13

It may be pointed out that these means are more reliable at the top of the table and less reliable at the bottom, judging from the probable errors of these figures.

It will be seen from this table that the average interval decreases vertically downwards and increases horizontally from left to right. The first decrease would be expected since it merely indicates that the more births there are in the period the smaller is the interval between births; but the second increase shows in all cases that the average interval increases from the first onwards for a given number of births. Thus when there are six births in the period the interval between the first and second births is less than that between the second and third, which in turn is less than that between the third and fourth, which is less than that between the fourth and fifth, which finally is less than that between the fifth and sixth. On the whole there is a gradual lengthening of the interval between births as more children are born.

The figures in Table IV were represented diagrammatically and it was seen that the points so obtained arranged themselves roughly in a definite manner. It was found that the arrays of

TABLE V.

No. of Births.	First Interval			Second Interval			Third Interval		
2	4·448	× 3	13·344						
3	3·320	× 4	13·280	*3*	4·313	× 4	17·252		
4	2·700	× 5	13·500	*4*	2·910	× 5	14·550	*4*	3·467 × 5	17·335
5	2·266	× 6	13·596	*5*	2·513	× 6	15·078	*5*	2·604 × 6	15·624
6	1·997	× 7	13·979	*6*	2·184	× 7	15·288	*6*	2·331 × 7	16·317
7	1·747	× 8	13·976	*7*	1·960	× 8	15·680	*7*	2·006 × 8	16·048
8	1·456	× 9	13·104	*8*	1·718	× 9	15·462	*8*	1·707 × 9	15·363
9	1·125	× 10	11·250	*9*	1·437	× 10	14·370	*9*	1·625 × 10	16·250
10	1·750	× 11	19·250	*10*	1·500	× 11	16·500	*10*	1·250 × 11	13·750

No. of Births.	Fourth Interval			Fifth Interval			Sixth Interval		
5	3·174	× 6	19·044						
6	2·387	× 7	16·709	*6*	2·747	× 7	19·229		
7	2·157	× 8	17·256	*7*	2·130	× 8	17·040	*7*	2·290 × 8	18·320
8	1·826	× 9	16·434	*8*	1·875	× 9	16·875	*8*	1·906 × 9	17·154
9	1·750	× 10	17·500	*9*	1·750	× 10	17·500	*9*	1·875 × 10	18·750
10	1·250	× 11	13·750	*10*	1·250	× 11	13·750	*10*	1·250 × 11	13·750

No. of Births.	Seventh Interval			Eighth Interval			Ninth Interval		
8	1·812	× 9	16·308						
9	1·875	× 10	18·750	*9*	1·875	× 10	18·750		
10	1·250	× 11	13·750	*10*	2·000	× 11	22·000	*10*	1·125 × 11	12·375

No. of Births.

points obtained by taking the vertical columns could be fitted reasonably by rectangular hyperbolae. This is brought out more easily in Table V where the corresponding figures are shown multiplied in each case by the integer which is in excess of the corresponding number of births by unity. Average values were taken and it was found that these arrays of points could be represented reasonably well by the curves:—

1st	Interval	$L=13{\cdot}50/(n+1)$
2nd	„	$L=15{\cdot}42/(n+1)$
3rd	„	$L=16{\cdot}71/(n+1)$
4th	„	$L=17{\cdot}55/(n+1)$
5th	„	$L=18{\cdot}08/(n+1)$
6th	„	$L=18{\cdot}36/(n+1)$
7th	„	$L=18{\cdot}52/(n+1)$
8th	„	$L=18{\cdot}61/(n+1)$
9th	„	$L=18{\cdot}66/(n+1)$,

where L is the length of the interval in years, and n is the number of births. Table VI shows (in italics) the values for the length of the interval obtained from these smooth curves; thus in the first column the figures are obtained by dividing 13·50 successively by 3, 4, 5, ... Side by side are reproduced the actual intervals for comparison. The diagram (Fig. 3) shows these smooth curves which indicate the gradual decrease in the length of the interval when we fix on one interval and trace the alterations in its length with the number of births, and the increase in the length of the interval when we concentrate on a definite number of births and trace the alterations from first to second, to third and so on intervals.

There appears then to be a natural law according to which the interval between births increases with the number of children born, and this, with the heredity factor affecting fecundity, inclines me to the opinion that the intervals between births are not random intervals.

TABLE VI.

Actual Mean Intervals and Smoothed Values (in italics).

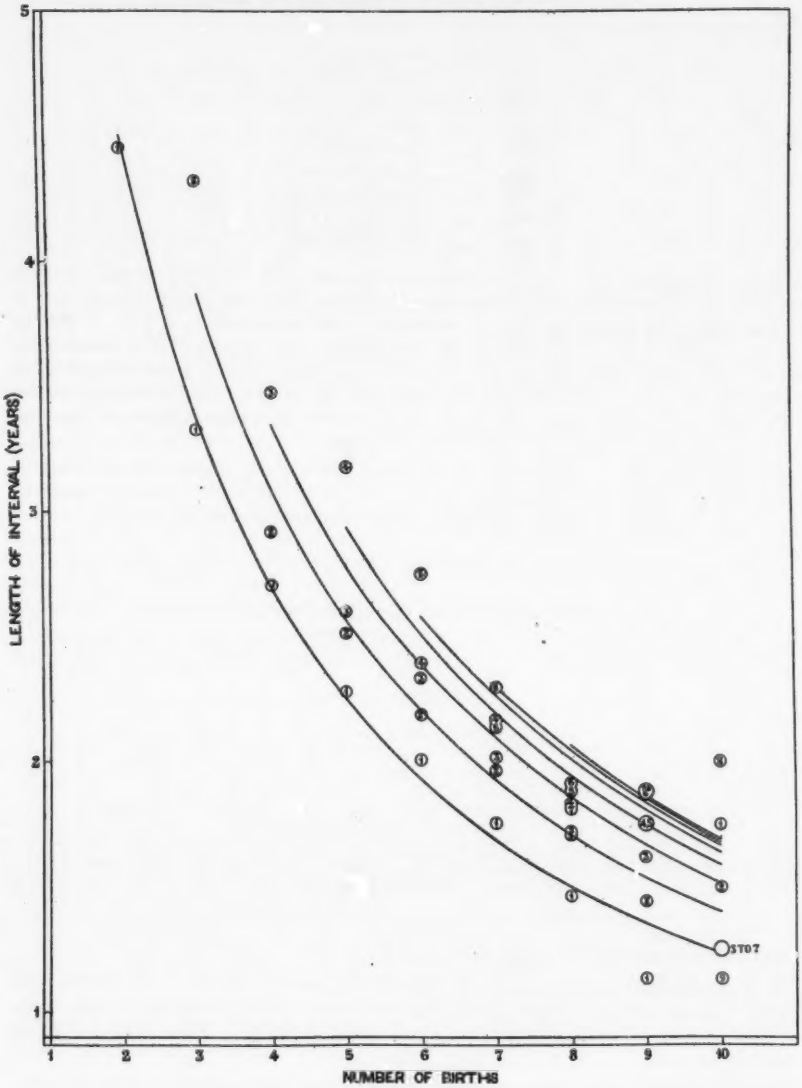
No. of Births.	1st		2nd		3rd		4th		5th		6th		7th		8th		9th	
2	4·45	*4·50*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
3	3·32	*3·37*	4·31	*3·86*	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4	2·70	*2·70*	2·91	*3·08*	3·47	*3·34*	—	—	—	—	—	—	—	—	—	—	—	—
5	2·27	*2·25*	2·51	*2·57*	2·60	*2·78*	3·17	*2·93*	—	—	—	—	—	—	—	—	—	—
6	2·00	*1·93*	2·18	*2·20*	2·33	*2·39*	2·39	*2·51*	2·75	*2·58*	—	—	—	—	—	—	—	—
7	1·75	*1·69*	1·96	*1·93*	2·01	*2·09*	2·16	*2·19*	2·13	*2·26*	2·29	*2·29*	—	—	—	—	—	—
8	1·46	*1·50*	1·72	*1·71*	1·71	*1·86*	1·83	*1·95*	1·88	*2·01*	1·91	*2·04*	1·81	*2·06*	—	—	—	—
9	1·13	*1·35*	1·44	*1·54*	1·62	*1·67*	1·75	*1·75*	1·75	*1·81*	1·87	*1·84*	1·88	*1·85*	1·88	*1·86*	—	—
10	1·75	*1·23*	1·50	*1·40*	1·25	*1·52*	1·25	*1·59*	1·25	*1·64*	1·25	*1·67*	1·25	*1·68*	2·00	*1·69*	1·13	*1·70*

(iii) *The Influence of the Sex of the Child.*

1. It was thought of interest to compare the interval following the birth of a male child, with that following the birth of a female child. This was done by obtaining the distribution of the lengths of intervals following male and female by grouping all the data. The results are shown in Table VII. on p. 421.

For comparison the percentages are shown in italics. These figures indicate, at sight, that there is no great difference between these distributions; and if we proceed further and apply the χ^2-test to these distributions (for this purpose grouping the numbers for the classes 9 years and onwards

FIG. III LENGTH OF INTERVAL WITH NUMBER OF BIRTHS & ORDER OF INTERVAL (SEE TABLE IV)



together), we get $\chi^2=9{\cdot}09$, and there being 18 classes this gives $P=.94$ This value of P, indicating that both of the above distributions may be considered as random samples from the same population, shows that the sex of a child has no influence on the length of the interval from its birth to the birth of the next child.

TABLE VII.

Length of Interval (years).	Interval after birth of Male		Interval after birth of Female	
Less than 1	22	1·3	18	1·1
1 —	152	8·8	158	10·0
1½ —	416	24·1	363	23·0
2 —	430	24·9	413	26·2
2½ —	236	13·7	199	12·6
3 —	144	8·4	118	7·5
3½ —	74	4·3	65	4·1
4 —	66	3·8	69	4·4
4½ —	46	2·7	49	3·1
5 —	35	2·0	28	1·8
5½ —	25	1·4	26	1·6
6 —	21	1·2	15	·9
6½ —	12	·7	14	·9
7 —	15	·9	8	·5
7½ —	9	·5	7	·4
8 —	7	·4	8	·5
8½ —	6	·3	6	·4
9 —	4	·2	3	·2
9½ —	1	·1	1	·1
10 —	1	·1	2	·1
10½ —	1	·1	3	·2
11 —	2	·1	3	·2
11½ —	1	·0	1	·1
12 —	—	—	1	·1
12½ —	—	—	—	—
13 —	1	·0	—	—
Totals	1727	—	1578	—

(v) Note on Professor Narumi's Paper. (See Vol. xv. p. 253.)

By Professor B. H. CAMP.

Through the courtesy of the Editor, I have been able to see the details of Professor Narumi's criticism of my proof. It is true that I should have, instead of the words "when $h=1$, etc.," (line 9, page 429) the following: "when $h\,Q(p)$ equals the value of s for which t equals 1." It then follows, by substitution in II, that: "for this special value of h which makes II greatest, $II=III$, etc." (line 14).

Professor Narumi has also suggested an alternative proof for this part of my lemma which, it seems to me, is susceptible of being put in rather simpler form than mine. It is, therefore, perhaps worth while reproducing here the entire proof with this modification.

Lemma. Let $Q(t)\geqq 0$, and be monotonic, decreasing, and let $d^2Q/dt^2\geqq 0$, in the interval $1\leqq t\leqq k$. Let the straight line $y=a+bt$ pass through the points of Q for which $t=1$ and $t=p$. Then

$$I\equiv\int_1^k t^{2r-1}\,Q(t)\,dt\geqq\int_1^k t^{2r-1}y\,dt\equiv II,$$

if $r\geqq 1$, $k\geqq 2$, $p=2rk/(2r+1)$.

It will be observed that, by subtracting a constant from each side of the inequality, the proposition can be reduced to the case where $Q(k)=0$. This being supposed done, let $h=Q(1)/Q(p)$, and note that by definition $p>1$, $h \geqq 1$. After determination of the values of a and b in accordance with the conditions imposed on y in the hypothesis, II becomes

$$II \equiv \frac{Q(p)}{p-1}\int_1^k t^{2r-1}(t-1)\,dt + \frac{hQ(p)}{p-1}\int_1^k t^{2r-1}(p-t)\,dt \quad \ldots\ldots(1).$$

Consider now the function represented by the line tangent to the curve Q at $t=p$, that is,

$$z = Q(p) + Q'(p)(t-p) \quad \ldots\ldots(2).$$

Since Q is concave upwards, it never crosses this tangent, and so $Q(t) \geqq z(t)$, and

$$I \equiv \int_1^k t^{2r-1} Q(t)\,dt \geqq \int_1^k t^{2r-1} z(t)\,dt \equiv III \quad \ldots\ldots(3).$$

In order to prove that $I \geqq II$, it is by (3) sufficient to show that $III \geqq II$. By the mean value theorem, $Q(p)-Q(1)=Q'(m)(p-1)$, i.e., $Q(p)+Q'(m)(1-p)=Q(1)=hQ(p)$, for some m such that $1 \leqq m \leqq p$, and therefore II may be written

$$II = Q(p)\int_1^k t^{2r-1}\,dt + Q'(m)\int_1^k t^{2r-1}(t-p)\,dt.$$

Comparing this with

$$III = Q(p)\int_1^k t^{2r-1}\,dt + Q'(p)\int_1^k t^{2r-1}(t-p)\,dt,$$

we see that $III \geqq II$ if $Q'(p) \geqq Q'(m)$, and if p is such that

$$\int_1^k t^{2r-1}(t-p)\,dt \geqq 0.$$

The first of these inequalities is easily seen to follow from the conditions that $p \geqq m$, and $dQ/dt \geqq 0$; and the second from the condition that $p=2rk/(2r+1)$.

A corrected set of tables follows :

Case I ($c=0$).

Formula	r	Factor
β_0	1	2·25
β_2	2	2·44
β_4	3	2·52

Formula	r	Factor
β_6	4	2·57
β_8	5	2·59
β_{10}	6	2·61

Case II ($c=1$)

r	θ			ϕ		
	$\lambda=2·5$	$\lambda=3·0$	$\lambda=3·5$	$\lambda=2·5$	$\lambda=3\,0$	$\lambda=3·5$
1	·015 6	·008 16	·004 81	·015 8	·008 23	·004 84
2	·001 40	·000 51	·000 22			
3	·000 15	·000 039	·000 012	ϕ equals θ to the number of places indicated		
4	$1·9 \cdot 10^{-5}$	$3·3 \cdot 10^{-6}$	$7·7 \cdot 10^{-7}$			

Case III ($c=2$).

r	$\theta/P(2s)$			ϕ		
	$\lambda=2{\cdot}5$	$\lambda=3{\cdot}0$	$\lambda=3{\cdot}5$	$\lambda=2{\cdot}5$	$\lambda=3{\cdot}0$	$\lambda=3{\cdot}5$
1	·275	·116	·0606	·379	·132	·065
2	·118	·0313	·0115	·134	·032	·012
3	·0561	·00985	·00263	·059	·010	·003
4	·0282	·00337	·00066	·029	·003	·001

Professor Narumi raises the question, as being of theoretical interest, whether the equality sign can hold in my formulae. He has shown that it can in his formulae, but in the practically important case when $k=\infty$ in his formula, page 246, his formula becomes identical with my fundamental Case I; which is in turn, as has been noted, identical with Meidel's formula.

CORRIGENDA. VOL. XV. PARTS I AND II.

Dr Oskar Anderson sends the following corrections to his paper "Über ein neues Verfahren bei Anwendung der 'Variate-Difference'-Methode," which appeared in our last issue: on p. 139, last line,

$$\textit{for}\ \frac{E\sum_{i=1}^{n-k-m}\Delta^k x_i^2}{\frac{(2k+2m)!}{(k+m)!\,(k+m)!}(n-k-m)}\quad\textit{read}\quad\frac{E\sum_{i=1}^{n-k-m}\Delta^{k+m}x_i^2}{\frac{(2k+2m)!}{(k+m)!\,(k+m)!}(n-k-m)},$$

and on p. 144 in the last line of the footnote *for* ρ_0' *read* ρ_j'.

INDEX

VOLUMES I TO XV

INDEX

VOLUMES I—XV

Abac
for determining probable error of correlation coefficients VII 411
— obstetric conjugate IX 539–40
Ability
inheritance of III 155
of schoolchildren V 119–29
Abruptness coefficients I 287
Corrections for moment-coefficients of limited range frequency curves XII 231–58
Actinosphaerium eichorni
Mass relations of nucleus and cytoplasm II 241–54
Acton, H. W. and W. F. Harvey
Altitude and number of erythrocytes VIII 280–91
Adler, J. E. and J. McIntosh
Albinism, histological examination of case of VII 237–43
referred to VII 244–47
Adoxa Moschatellina
Variation in II 108–13, VII 218–22
Afghans
head measurements IV 293
Africa
congenital anomalies in Native African race X 1–24
pigmentation anomalies among Nyasaland natives IX 333–65
Age
brain-weight and IV 51–58, 109, 118–23
hair- and eye-colour and age in man III 462–66
intelligence and; Pearson's Data V 112–19
— Drinkwater's Data XIV 396–98
marriage, age at II 20–23
— — and duration of life IV 256–63
— — and duration of marriage and number of offspring IV 264–80
— — and fertility XIV 337–54
reaction times and sensory acuity XV 346–72
stature and; Australian criminals I 38–49
— Glasgow schoolchildren X 288–339
susceptibility to smallpox and IV 313–31, 483–504
variability and, in snails V 387–99
weight and, Glasgow schoolchildren X 288–339
Age and Area
and Natural Selection (Willis' Theory) XV 89–108
Ainos
Cranial measurements of II 346–56, IV 298
— — compared with English, Whitechapel III 206, 221–26, 234–39
— — — Moriori XI 95, 98, 102, 132
— — — Naqada I 424–65
— — — Negro VIII 302–14, 321–24
— — — Tibetan XIV 206–26
— — nasal characters IX 395, 399, 403–7, 421, 425, 427, 429, 432
Head measurements of IV 165, 293
Air Force Data
inter-relations of vital capacity, stature, stem length and weight XIV 326–36
Albinism
Deaf mute, Albino VII 247
Histological examination of case of (with plates) VII 237–43
— — of eye of human albino (with plates) XIII 46–51
Internal VII 244–47
and Mendel's Laws III 107–8
— — Note on III 471–72
— — Further correction IV 231
Mice II 101–4, 165–73, 282–85
— Albino and Grey, crosses of IV 1–12
— Albino and Waltzing, crosses of III 1–51
Nyasaland natives, pigmentation anomalies among IX 333–65
White-Lock Family, Barton XIII 347–49
See also Piebalds
Altbayerisch
See Bavarians
Alternative Inheritance in peas I 228–54
See also Mendelism, Coat-, Hair- and Eye-colour
Altitude
and number of erythrocytes VIII 280–91
American Bladder Nut (Staphylea)
selective elimination during development of fruits VII 452–504
American Indian
variation in female pelvis IX 34–57
correlation of measurements of pelvis IX 486–529
Amphibia
sesamoids in XIII 391
Anaemia
and altitude VIII 280–91
Ancestral Correlation, see Inheritance
Ancestral Inheritance, Law of
II 211–40, III 255–57, 263, VIII 239–43
and De Vries' Mutation Theory I 370–73
and Mendelism III 109–12
and Prof. Ziegler V 184
in rabbits II 300–6
spinax III 342–45, 363–65
Anderson, O.
Variate Difference Correlation Method X 269–79, XV 134–49
— — — — corrigenda XV 423
— — — — referred to X 340–46, 355, 489, 496, 499, XI 136, XIV 285, 289, 290
Anemone nemorosa
Variation of number of sepals I 307–9
Angoni, Negro
Cranial measurements and comparison with other races VIII 292–337
Annulata
Variation and Correlation in Earthworm IV 213–29
Anomalies
pigmentation, Nyasaland Natives IX 333–65
See also Albinism, Piebald; Deformities

Anthriscus sylvestris XI 44–47
Anthropometry
Australian Data I 30–49
Cairo-born natives XI 67–81
Criminals I 177–227, XI 1–37
— head measurements of III 60–62
Egyptians (modern) VIII 66–78
— — corrigenda IX 332
Inhabitants of Eastern Mediterranean XIII 92–112
Oxford Anthropometric Laboratory VIII 40–51
Russian XV 254–70
Scotland, inmates of asylums V 298–350 and supplement
Sexing osteometric material X 479–87
Stature span and forearm in families II 357–462
See also Head measurements, Stature, Weight
Anticipation
Law of and insanity X 358–83
Apes
study of nasal bridge in anthropoid apes and relation to that in man IX 391–445
Aphis (Hyalopterus trirhodus)
variation and parthenogenetic inheritance in I 129–54
Apis mellifica (bee)
comparison of variability of wings with those of wasps V 419–21
Arabs
cranial measurements of IV 298
head measurements of IV 293
Arcella
variation and correlation in II 321–37
Arcoleo, G. (reference)
Mendelism and Albinism III 107–8, 471, 472, IV 231
Armenians
Head measurements of IV 165, 293
Arnica montana
Variation in I 26–27
Arrays
probable error of mean of, found from regression line IX 9, 10
— — and standard deviation of and frequency distribution of mean of array XI 239–61, XII 267–81
Arthropoda I 129–54, II 191–210, 255–59, 307–20, III 58, 59, 113–30, V 14, 16, 17, 19, 26, 365–86, 407–22, VI 103–5, 311–26, 329–47, VII 48–63, VIII 1–12, IX 201–7
Artistic Capacity
Association of with steadiness and rapidity of hand XII 170–77
Assertiveness
See Shyness
Association
of attributes in statistics II 121–34
coefficient of VIII 109–22, IX 159–316
surface of constant IX 183, 184, 534–37
on Theories of IX 159–316
Assortative-mating
among the deaf IV 473
in conjugation of Paramecium V 249–63
Assortative-mating (*cont.*)
in man II 372–77, 397, 481–98
Aster I 311, II 113, 114
Asylums in Scotland
anthropometry of inmates of V 298–350 and supplement
Asymmetry
in crabs II 307–20
Athletic Power
inheritance of III 152–55, 182
intelligence and V 126–31, 133, 144
Aurelia aurita (jellyfish)
variation in I 90–108
Australians
anthropometric data I 30–49
cranial measurements IV 297, 298
— — compared with Moriori XI 95, 96
Fertility, duration of life, etc., among IV 233–85

Bacillus coli
intermediates in, and selection IX 331–32
Barometric Heights
Skew correlation surface XIV 371–77
Barrington, A., A. Lee and K. Pearson
inheritance of coat-colour in greyhound III 245–98
Barrington, A. and K. Pearson
inheritance of coat-colour in cattle IV 427–64
Bartolinus, T. (reference)
sesamoids XIII 147, 357
Barton, E. R. (reference)
Whitechapel crania III 194, 196
White-lock family XIII 347–49
Bateson, W.
Albinism in Sicily, a correction III 471–72
— — further correction IV 231
(reference) Biology, Pearson's reply to criticism I 320–44
— Mendelism II 44–55
— Mendel's theory of Heredity II 286–98
— Mice-breeding III 15–18
Bauhin, C.
(reference) sesamoids XIII 146
Bavarians
Brain-weights and other anthropometric characters IV 13–104, 133
Cranial measurements II 346, 350, IV 297, 298
— — compared with English, Whitechapel III 206–15, 221–26
— — — Moriori XI 95–98
— — — Naqada I 424–65
— — — Negro VIII 302–14, 321–24
Bayes
(reference) problem of influence of past experience on future expectation XIII 1–16
— — — — — — — note XIII 300–1
— — — — — — — See also IX 73, 74
Bean
inheritance in II 290–92, 499–503
weight of seed and characteristics of plant IX 11–21, X 72–84

Beaven, E. S.
"Square-yard" plots for testing cereals XV 278–85
Beddoe, J. (reference)
measurement of capacity from cranial circumferences III 366–97
Bee
average length of absence from nest V 382–86
comparison of variability of wings with those of wasps V 419–21
Beech tree
homotyposis and differentiation in leaves (with plates) III 104–7
Beetle (Carabus auratus)
relative variability of sexes in VI 103–5
Beeton, M. and K. Pearson
inheritance of duration of life I 50–89
Behr, K. von (reference)
inheritance of sex in royalty V 73–75
Bell, J.
Pulse and breathing rates VIII 232–36
Table of the Mesodacryal Indices VIII 338–39
Bell, J., W. Rowan and K. M. Parker
Terns' eggs X 144–68
Bell, J. and K. V. Ryley
Study of Nasal Bridge in Anthropoid Apes IX 391–445
Bell, J., E. Y. Thomson and K. Pearson
Wasps VII 48–63, VIII 1–12
Benington, R. Crewdson
Crania, Negro, with special reference to Congo and Gaboon VIII 292–337
Cranial Type contours VIII 123–201
— — — (reference) XI 86, 90, 99, 101, 103, 107, XIV 228, 230, 232, 234, 241
Bentham, Jeremy
Brain-weight and head measurements of IV 144, 145, 148
head measurements and determination of capacity III 393–96
plates of auto-icon of to face III 393
Bergström, S.
Product-moments, normal correlation surface of n-variables XII 177–83
Bessels, E. (reference)
Crania XIV 221
Beta
probable error of β_1 and β_2, Tables to facilitate computation VII 128–47
values of β-constants for distributions of moment-coefficients of any order in samples XII 260–66
Beta-Function
incomplete, expansion of in tetrachoric functions XIV 169–75, 181–85
Bibliography
1904–6, V 210–12. 1904–7, V 482–86
1906–7, VI 123–28. 1905–8, VI 444–51
1907–9, VII 413–20. 1908–12, IX 140–58
Bimodality
of frequency curves III 84–98
— — polygons, plant data I 22
Binary Fission
relation to Variation I 400–7
Binet-Simon Tests
inheritance of intelligence XIV 386–405
— — comparison with method of broad categories XII 367–72
Binomial
Exponential Limit to, Tables X 25–35
Frequency Constants, Probable Error of X 37–38
Note on moments about the mean XV 410–12
Poisson-Exponential and Gaussian, comparison of for all frequency variations X 41–44
Series, components of compound frequency distributions XI 139–44
Biology
Fundamental conceptions of I 320–44
Biometrika
Foundation of V 35
scope of I 1, 2
spirit of I 3–6
Biometry I 7–10
Francis Galton on I 7
History of in connection with Weldon V 13–52
Birds I 164–76, 256–57, IV 363–73, VI 420–32, VII 1–42, 234, 236, 398–409, VIII 435, 436, IX 458–60, 538, 539, X 144–68, XII 308–54
also given under names of species
sesamoids in XIII 387, 388
Births
time interval between XV 412–21
Bischoff, T. L. W. (reference)
Brain-weight IV 16
Bladder-Nut (Staphylea)
Selective elimination during development of fruits VII 452–504
Blakeman, J.
Brain-weight and head measurements IV 124–60
— — (reference) IX 478–80
Test for linearity of regression IV 332–50
Blakeman, J. and K. Pearson
Probable error of mean square contingency V 191–97
Blanchard, N.
inheritance in horses I 361–64, II 229–34
Blood corpuscles
in tadpoles VI 402–19
Bloodroot (Sanguinaria canadensis)
morphology of fruit of VII 305–51
Boas, Franz (reference)
Association, theories of IX 159–316
cranial measurements I 410
Bohemians
Brain-weights and other anthropometric characters IV 13–104, 133
Bombus (bumble bee) V 382–86
Bonney, W. V. (reference)
sesamoids XIII 378
Boole, G. (reference)
problem of inverse probabilities XIII 2, 3
quadrature formula I 275
Born, G. (reference)
Frog's egg V 161

Borneo
cranial measurements, nasal characters IX 395, 399–407, 410–21, 425, 427, 432
Boyd (reference)
brain-weights IV 16–18
Brachydactyly
inheritance of, note on VI 327
Nyasaland natives (with plates) X 17, 18
Brain-weight
age and IV 51–58
head measurement and IV 105–22, 124–60
— — prediction formulae IV 146–49
skull measurement and IV 67–74
stature and IV 58–64
variation and correlation in IV 13–104
weight and IV 64–66
weights of viscera and IX 478–85
Bravais, A. (reference)
History of Correlation, Normal Surface XIII 28–32
Breathing
association with physique and intelligence in criminals XI 1–19
and pulse rates and stature VIII 232–36
Breeding
See Inheritance
Brierley, J. (reference)
crania XIV 221, 222
Brindley, H. H. (reference)
regeneration of feet in Blattidae VI 311–26
British
hair- and eye-colour correlations III 461
eye-colour and age III 463, 464
See also English, Scottish
Brittle-Star
(Ophiocoma nigra) II 463
Broad Categories
comparison of method with series measured by Binet-Simon test XII 367–72
influence of on correlation, class-index correction IX 116–39, 316–20, XIV 261–80
methods of correlation for variates grouped in; see Correlation coefficient, Tetrachoric and Polychoric, Correlation Ratio
probable error of class-index correction XIV 264–80
probable error of distance of mean of group from mean of sample XIV 262–64
probable errors of frequency constants of a system grouped in XIII 128–32
Broesike, G. (reference)
Crania, Moriori XIII 339
Broncho-pneumonia V 428, 429
Brood-mares
thoroughbred, distribution of fecundity in I 289
Brown, J. W. and M. Greenwood
weight of human viscera IX 473–85
Brown, W.
Mathematical Intelligence VII 352–67
Browne, E. T.
Variation in Aurelia aurita I 90–108
Browne, Sir Thomas
skull, portraits and ancestry (with plates) XV 1–76
Brownlee, John
correlation coefficient and Mendelian distributions (reference) VIII 420–24
death-rates IX 537
severity of cases of disease, means of estimation of V 423–35
smallpox and vaccination IV 313–31
Bruce, D. (reference)
Trypanosome strains X 89–143
Buist, J. M.
Anti-typhoid inoculation VI 366–75
Bulloch, W. (reference)
Phagocytic counts, opsonic index VI 378, 379
Burmese
Crania, measurements, contours and comparison with other races (with plates) XIII 176–262
Burnside, W. (reference)
Criticism of Burnside's suggested modification in formula for distribution of means XV 401–6
Butter Fat, see Cows
Buxton, L. H. Dudley
Inhabitants of Eastern Mediterranean (with plate) XIII 92–112
Byrne, L. W.
Bony plates of John Dory II 115–20

Cairo natives
anthropometric measurements XI 67–81
Cambridge Graduates
Intelligence and head measurements V 106–25
Camp, B. H.
Note on Narumi's paper on limits to tails of frequency curves XV 421–23
Camper, P. (reference)
sesamoids XIII 151–53
Campion II 47–55
Canary (Breeding)
— Davenport VII 398–409
— Galloway (with plates) VII 1–42, 398–403
— Heron VII 404–9
hybridisation of, Galloway VIII 435–36
Cancer
Death-rate VII 276–304
correlation between corrected cancer and diabetes death-rates XI 191–200
and diabetes VII 284–95
and insanity VII 290–95
Cancer-houses VIII 430–35
See also VIII 396–403, 404–12, IX 31–33
Capacity, cranial
Correlation with cephalic index and intelligence III 374, 392–96
determination of I 419–22, III 203–6, V 94, XI 83, 84, XIII 178, 179
formulae for determination of from external measurements X 188–93
measurement of from cranial circumferences III 366–97
Carabus auratus (beetle)
relative variability of sexes in VI 103–5

Carnivores
sesamoids in XIII 351, 352
Casserius, J. (reference)
sesamoids XIII 147
Castle, W. E. (reference)
Law of ancestral inheritance III 109–12
Castrel, D. B. (reference)
Bees, variability in V 419–21
Cattle
inheritance of coat-colour in IV 427–64
See also Cows
Cave, B. M. and E. M. Elderton
Probable error of biserial η, table to assist computation of XI 302
Cave, B. M. and K. Pearson
Variate Difference Method, illustrations X 340–55
Cave, B. M., H. E. Soper, A. W. Young, A. Lee and K. Pearson
Small samples, distribution of correlation coefficient in XI 328–413
Celandine
Change in organic correlation during flowering season I 125–28
Variation and correlation in II 145–64
— in number of sepals and petals I 11–20
Cell-frequencies
Probable errors of X 40–47
Cell size
and body size in Daphnia magna II 255–59
Cephalic Index
of schoolchildren, inheritance of III 154
reduction to cranial index VIII 68–70
See also Cranial measurements and Head measurements
Cercis canadensis
homotypic correlation of fertility characters XI 201–14
Cereals
testing varieties of XV 271–93
Cerebrum
weight of IV 74–76
di Cesnola, A. P.
colour in Mantis religiosa III 58, 59
snails, natural selection in V 387–99
Charlier, C. V. L. (reference)
Notice of papers on Law of Errors V 206–9
Chatham Islands
aborigines of, see Moriori
Cheselden, W. (reference)
sesamoids XIII 150–1
Chicory
Variation and homotyposis V 184–89
Chilomonas
Variation in V 53–72
Chinese
cranial measurements IV 297, 298
— —.compared with Burmese XIII 235–47, 250, 251
— — — Tibetan XIV 206–26
head measurements IV 294
Chiroptera
sesamoids in XIII 355
Choice
in distribution of observations XII 1–85
Chromatin
mass relation of chromatin and cytoplasm II 241–54
Chrysanthemum leucanthemum
variation in I 309–15, 319
Cichorium intybus, L.
variation and homotyposis in V 184–89
Claremont, C. A.
Correlation between corrected Cancer and Diabetes Death-rates XI 191–200
Class-Index
correlation of a variate with its IX 116, 117, 124
— — probable error of XIV 261–80
Class-Index Correction IX 116–39
See also IX 316–20
Probable error of XIV 261–80
Clausilia itala (snails)
selection in III 299–306
Clausilia laminata (snails)
natural selection in I 109–24
Clendinning (reference) III 63, 68
Brain-weights IV 31, 129, 133
Climate
and cancer death-rate VII 283
and Fertility I 32–33
Clinograph III 469
Clover
experiments in growing I 371–73
Coat-colour
inheritance of, in cattle IV 427–64
— in greyhound III 245–98
— in horses I 361–64
— in mice II 101–4, 165–73, 282–85, 294–98, III 1–51, IV 1–12 (with plates)
— in rabbits II 299–306
— in racehorses II 229–34
Cobb, J. A.
Correlation coefficient, effect of errors of observation on (sex-ratio) VI 109
— — — — (reference) VI 109–11, 121, 122
Cockayne, E. A.
Piebald Family X 197–200
Cockroach (Blattidae)
Four-jointed tarsus and regeneration of feet in VI 311–26
Coelenterata I 90–108, 255, VII 373–81
Colligation
Coefficient of IX 291–94
Colour
protective value of in Mantis religiosa III 58–59
See also Coat-, Hair- and Eye-colour and Pigmentation
Composite Photograph
Russian Types XV 270
Compound Frequency Distributions
in which components can be described by Binomial series XI 139–44
resolution of into component parts (Trypanosome strains) X 123–42
Compound variable
Frequency constants of, determined from constants of components VI 437–38
See also Index distributions

Congo
Crania, measurements and comparison with other races (with plates) VIII 292–337
— compared with Moriori XI 95–112, 125–32
— — Tibetan XIV 208–47
— nasal characters IX 394–437
Conjugation
in Paramecium V 213–97
Conscientiousness
inheritance of III 152–55, 187
intelligence and V 126–31, 134, 136, 146
Contingency
class-index correction, application of to IX 128–38
comparison with Yule's methods IX 180–83, 216–56
criticism of method V 472, 473
multiple, general theory of, with special reference to partial contingency XI 145–58
application to the question of significant difference of corrected death-rates XI 155–58, 163–84
Contingency, Coefficient of Mean Square
Mean value of XI 216
Probable error of V 191–97, X 570–73
— — without approximation XI 215–30
— — summary of formulae XI 229–30
— — corrigenda XII 259–60
Standard deviation of XI 220–22
Test for zero contingency XI 227, 228
Cool, C. and A. N. Koopmans
Umbelliferae, variation and correlation of number of umbel rays XI 38–49
Copts
cranial measurements, compared with Naqada I 425–65
— — — Negro VIII 302–14, 321
Correction
Class-Index IX 116–39, 316–20
— — Probable Error of XIV 261–80
for correlation ratio VIII 254–56, XIV 412–17
of death-rates for age distribution XI 159–60
for moments, Sheppard III 308–12, V 450–59
for moment-coefficients, Abruptness XII 231–58
Correlation
between a variable and the deviation of a dependent variable from its probable value VI 438–43
between the deviation in a product moment about a fixed axis and one about an axis through the centroid IX 6–9
danger of some formulae VIII 109–22
effect of changing scale order on V 176–78
History of, notes on XIII 25–45
influence of broad categories on IX 116–39
influence of selection on VIII 437–43
influence of double selection on variation and correlation of two characters VI 111, 112
Intra-class, spurious values X 412–16
normal, see Normal Correlation
Pseudo Ranks method of Yule IX 226–56
Correlation (*cont.*)
Ranks and Grades Method, extension, relation between actual variate correlation and correlation as determined from ranks or grades X 416–18
— — — correction for ties, experimental determination of probable error XIII 264–82
— — — application to Examination results XIII 302–5
Skew surface XIV 355–77
spurious, elimination of X 179–80
See also Variate Difference Method
spurious X 412–16
tetrachoric functions, tables VII 437–51
— — supplementary tables for high correlations VIII 385–95
— — further supplementary tables for high correlations XI 284–91
tetrachoric, probable error of IX 22–27
"theoretical value" VIII 110–22, IX 166–71
Variate Difference Method, see Variate Difference Method
Correlation Coefficient
Application of to Mendelian Distribution VIII 420–24
Biserial Expression for VII 96–105
— — Probable error of X 384–90
effect of errors of observation on VI 109, XII 357–59
Integral variates, method of calculation for VII 214–18
inter- and intra-class, calculation of IX 446–72
Polychoric Table, enneachoric method XII 95–106
— — polychoric method, from weighted means of tetrachoric values XII 106–19
— — comparative results of various methods XII 121–23
— — method of fitting best Normal Surface XIV 127–56
— — approximate Method used in practice XIV 152–56
Probable Error of IX 4–6
— — abac for determining VII 410, 411
— — small samples IX 91–115
See also Probable error
Small samples, Frequency distribution of, in Experimental work and empirical formula (Student) VI 302–10
Work to a second approximation (Soper) IX 91–115
Actual Frequency distribution (Fisher) X 507–21, XI 328–413
Correlation, Multiple VIII 437–43
errors with regard to X 181–87
generalised probable error in multiple normal correlation VI 59–68
Hyperspherical Goniometry and XV 173–208
and partial, properties of relations between coefficients of different orders XI 231–38
Correlation, Partial VIII 442–43
Partial correlation ratio X 391–411, XI 50–66

Correlation, Partial (*cont.*)
probable error of partial correlation coefficient VII 411–12
and multiple, properties of relations between coefficients of different orders XI 231–38
Correlation Ratio
Bi-serial and tri-serial methods VII 248–57
Correction for grouping IX 316–20
See also IX 116–39
correction for, comparison with mean value for zero correlation VIII 254–56
correction for XIV 412–17
Partial. Part I. Theoretical X 391–411. Corrigenda, X vi
— Part II. Numerical XI 50–66
Probable error of biserial XI 292–301
— — table to assist computation of XI 302
Cows
inheritance in production of butter fat VII 106–26
— of quantity and quality of milk VII 548–50
Crabs II 191–210, 307–20, V 19, 26
Craig, J. I.
Anthropometry of Egyptians VIII 66–78. Corrigenda, IX 332
Crampton, H. E.
Variation and Elimination in Lepidoptera III 113–30
Crangon vulgaris V 14, 16, 17
Crania
anomalies of (with plates) Whitechapel III 217–21
— — Burmese XIII 232–35
— — inheritance of XV 406–10
— — Moorfields V 100–3
— — Moriori XI 134, 135
— — Negro VIII 322–33
— — Tibetan XIV 201–3
See also "Remarks" on tables of individual measurements
Anomalies of, Artificial deformation XIII 338–46
— Persistence of synchondrosis condylo-squamosa (with plates) VIII 257–61
— Pigmy crania (with plates) VIII 262–66
Asymmetry of XIII 251–59
Capacity, determination of I 419–22, III 203–6, V 94, XI 83, 84, XIII 178, 179
— formulae for determination of from external measurements X 188–93
— Measurement of from circumferences III 366–97
Location of certain Cranial "points" I 413, XIII 180–85, XIV 200, 201
Mandibles, measurements of, English, Whitechapel III 213. Table VIII to face 244
— — Moriori XI 132, 133. Table to face 134
— — Naqada I 417, 426, 427. Table to face 466
— — Negro VIII 298–300, 308. Tables III A and B and IV to face 336
— — Tibetan XIV 253–60
Measurements and methods of measurement I 412–19, III 199–206, V 91–94, VIII 296, 297, XI 83–86, XIII 177–83, XIV 196–200

Crania (*cont.*)
Mesodacryal Indices, discussion of use of in racial characterisation, etc. VIII 297, 315–20, IX 391–445, XIII 246, 247
— — Tables of VIII 338, 339, IX 401, 402
Methods of measurement and apparatus (with plates) I 412–19
Nasal Bridge, study of in Anthropoid Apes and relation to that in man IX 391–445
— — study of in Congo and Gaboon crania VIII 315–20
Occipital Bone, observations on VIII 257–61
— Index, Tables for calculating XIII 261–62
Races. Burmese, measurements, contours and comparison with other races (with plates) XIII 176–262
— Egyptian, measurements and description of Pigmy crania (with plates) VIII 262–66
— — observations on occipital bone with special reference to persistence of synchondrosis condylo-squamosa (with plates) VIII 257–61
— English, Moorfields, variation and correlation in and comparison with Whitechapel (with plates) V 86–104
— — Whitechapel, variation and correlation in and comparison with other races (with plates) III 191–244
— Long Barrow and Round Barrow, measurements and descriptive catalogue IV 351–62
— Moriori, measurements, contours and comparison with other races (with plates) XI 82–135
— — question of artificial deformation of (with plates) XIII 338–46
— Naqada, variation and correlation in (with plates) I 408–67
See also II 345–47, 504–8 on Homogeneity in Collections of Crania
— Negro, Congo-Gaboon, measurements and comparison with other races (with plates) VIII 292–337
— Tibetan, measurements, comparison with other races, contours, etc. (with plates) XIV 193–260
See also under names of various races for comparative measurements
Simotic Index, discussion of and use of in racial characterisation VIII 317–20, IX 391–445, XIII 246, 247
Tables of Individual Measurements, Burmese XIII to face 262
— — — English, Moorfields V „ 104
— — — — Whitechapel III „ 244
— — — Moriori XI „ 134
— — — Naqada I „ 466
— — — Negro VIII „ 336
— — — Tibetan XIV „ 260
Type Contours, comparison of direct measurements with contour values (Moriori) XI 125–32, (Burmese) XIII 210–17, (Tibetan) XIV 249–52
— — comparison of for Modern English and Whitechapel VIII 131–33

Crania (*cont.*)
Type Contours, comparison of English and other races VIII 131–38
— — — Moriori and other races XI 103–12
— — — the three types of Burmese XIII 223–31
— — — Tibetan and other races XIV 227–48
— — construction of VIII 123–29
— — XI 86–89, XIII 185–210, XIV 227, 234, 239
— — plates VIII 139–201, XI 113–28, XIII 188–93, 195–200, 203–208, XIV 229, 230, 235, 236, 243, 244
Cranial measurements
and brain-weight IV 67–74
discussion of frequency distributions of I 441–54, III 226–31
various races IV 297, 298
Cranial Types II 508–12
Craniological Notes II 338–56, 504–12
Criminal Anthropometry XI 1–37
Criminality
and Education in Egypt X 280–83
Criminals
Anthropometry and Identification of I 177–227
Australian, stature, etc. I 38–49
Egyptians VIII 66–78, XI 67–81
Head measurements of III 60–62
Criminological Works
reviews of VII 231–33
Cripps, L. D., M. Greenwood and E. M. Newbold
Inter-relation of vital capacity, stature, weight, etc. XIV 316–36
Cro-magnon
cranial contours, plates VIII 149–53
— — compared with Moriori XI 103–12
Cruciferae
Statistical study of X 215–68
Cuckoo
eggs of, dimensions, coloration, relation to egg of foster-parent, etc. I 164–76, IV 363–73
— test of existence of sub-races IX 458–60
Cuénot, L. (reference)
Mice-breeding IV 1–3
Curve fitting I 265–303, II 1–23
Abruptness Coefficients, corrections for moments of limited range frequency curves XII 231–58
Psychophysical data XII 216–30
See also Frequency distributions, Goodness of fit
Curves
Method of drawing to various scales III 469–71
Cytoplasm
Mass relation of nucleus and cytoplasm II 241–54
Czekanowski, J. (reference)
Notice of paper on head and skull measurements VI 123
— — correlation between various muscles VI 123
Dante
Brain-weight and head measurements of IV 147–48
Daphnia magna
parthenogenetic generations of I 129
relationship between size of cell and size of body II 255–59
Darbishire, A. D.
Mice-breeding II 101–4, 165–73; 282–85, III 1–51
— (reference) II 294–98
Darwin, C. (reference) I 1–6, III 400, 401, 424, 447, 449
Darwin, L. (reference)
environment and heredity X 183–87
Darwinism
Biometry and some recent Biology VII 368–85
Davenport, C. B.
canaries, inheritance in VII 393–400
— — (reference) VII 1–5, 401–3, 403–9
jellyfish, variation and fertility in I 255, 256
Pectinatella magnifica, statoblasts of I 128
Davenport, E. (reference)
Principles of Breeding, Notice of book on VI 444
Davin, A. G. and K. Pearson
Sesamoids of the knee-joint XIII 133–72
— — — evolution of XIII 350–400
Davis, J. Barnard (reference)
Crania III 192, 377, 378, 380–88
Deafness
inheritance of IV 465–82
Death
age at, during first year of life IV 510–16
Death-rates
correlation of death-rates in infancy and childhood X 488–506, XIV 300–8
— corrected cancer and diabetes death-rates XI 191–200
corrected XI 159–60, 191–200
criteria for the existence of differential death-rates XI 159–84
diphtheria, effect of isolation on attack and death-rate X 549–69
local, suggestion for work on I 384
note on IX 537
phthisis, check to fall of XII 374–76
pigmentation and death-rate, schoolchildren in Scotland VI 187
selective, in Man IX 58–68
Deformities
Brachydactyly VI 327, X 17, 18
Digital anomaly XIV 410–11
Four-jointed tarsus in Blattidae VI 311–26
in Nyasaland natives X 1–24
Polydactyly X 18, 19
Split-hand and split-foot VI 26–58, 69–79, 117–18, IX 381–90, 540
Degeneracy
effect of differential fertility on VII 258–75
Delinquents
Study of Women, in New York, Review XIII 305–8

Deniker, J. (reference)
Anthropometry of inhabitants of Eastern Mediterranean XIII 94
Table of Anthropometric measurements IV 296
Derry, D. E. (reference)
sexing of crania IX 394, 396
Deviates
of Normal Curve, Table of V 405
Diabetes
Age distribution of deaths VIII 225–31
Cancer and VII 284–95
Correlation between cancer and diabetes death-rates XI 191–200
Dietaries
Statistical Study of IX 530–33, X 169–72, 172–74
Difference Method of Correlation, see Variate Difference Method
Difference Problem
Galton I 385–99
Differentiation I 321–26, 334–44
and homotyposis in beech leaves III 104–7
Digital Anomaly
unusual case of (with plates) XIV 410–11
See also Brachydactyly, Polydactyly, Split-foot
Digitalis gloxiniaeflora (foxglove)
Breeding experiments with XI 303–27
inheritance in and result of selective breeding XIV 103–26
Dimon, A. C.
Study of Nassa obsoleta and Nassa trivittata II 24–43
Dimorphic frequency curves III 84–98
resolution of IV 230, 231
See also Compound Frequency Distributions
Diphtheria
Isolation, effect of on attack and death-rates X 549–69
pigmentation and VIII 24–39, 360–65
Discontinuity of variation I 326–44
Disease
class differentiation in mortality from different diseases XV 382–400
distribution of cases in houses VIII 396–403, 404–12, 430–36, IX 28–33; see also VIII 413–19
immunity IV 313–31, 483–510, V 423–34
means of estimating severity of cases V 423–34
See also under separate headings
Divergence of Classes V 198–203
Coefficient for measuring V 334–38
Dogs
colour in, compared with cattle IV 451–54
inheritance of coat-colour in III 245–98
Dominance I 228–54, II 44–55, 211–15, 228
coat-colour in mice II 101–4, 165–73, 282–85, III 1–51
— in rabbit II 299–306
Mendelian, Jew and Gentiles VIII 426, 427
See also Mendelism
Donaldson, H. H. (reference)
brain-weights IV 15, 18, 65, 78, 80
Donkin, H. B.
head measurements of criminals (introductory note) III 60
Doodson, A. T.
Frequency curves, relation of mean, median and mode XI 425–29
Dravidians, see Maravar
Drawing
association with other capacities in school-children VII 222–26
Dreyer, G. (reference)
Vital capacity, height, weight, etc. XIV 325–36
Driesch, H. (reference)
morphogenesis V 53, 70–72
Drinkwater, H. (reference)
Inheritance of intelligence XIV 394–405
Duckworth, W. H. L. (reference)
Crania XI 82, 91, 97, XIII 339–44, XIV 220
Duffell, J. H.
Gamma Function Table VII 43–47
Dunbar, F. J. and R. Pearl
Variation and correlation in Arcella II 321–37
Duncker, C.
Asymmetry in crabs II 307–20
Duncker, G.
β_1 and β_2, high values of VIII 238
Duration of Life
Assortative-mating among English II 481–98
correlation with age at marriage IV 256–63
— — number of offspring I 34, IV 233–47
inheritance of among English I 50–89
mean, for individuals dying in first year of life IV 510–16
and fertility I 34–38, IV 233–47
Dwight, T.
Method of sexing X 209–12

Earthworm
Variation and correlation in IV 213–29
Echinodermata II 463–73, VI 236–301
Edentates
sesamoids in XIII 366
Edgeworth, F. Y.
Frequency distributions (reference) IV 199–204, V 168–71
Goodness of fit (reference) XI 418–25
History of Correlation (reference) XIII 42–44
notice of papers on Law of Error (reference) V 206–9
problem of inverse probabilities (reference) XIII 4
wasps and bees V 365–86
Editorial I–61
Gamma Function, Formula for determining $\Gamma(x+1)$ VI 118–19
Normal curve, table of ordinates of XIII 426–28

Editorial (*cont.*)
"Peccavimus," Probable Error of Contingency coefficient without approximation, corrigenda XII 259–60
— Conditions under which Probable Errors of Frequency Distributions have a real significance, corrections and additions XII 260–66
— Probable Error and Frequency distribution of errors in mean and standard deviation of arrays XII 267–81
— Corrigenda to 2nd supplement to memoir on skew variation XII 281
Polymorphism in plants I 304–6
Probable Error of frequency constants, Pt I (1 variate) II 273–81
— — — — — Pt II (2 variates) IX 1–10
— — — — — Pt III, Grades Method XIII 113–32
— — of Mendelian Class frequency XI 429–32
Sheppard's corrections, elementary proof of III 308–12
Small samples, distribution of standard deviations of X 522–29
Education
influence of, in Egypt X 280–87
Edwards, C. L.
Variation, development and growth in Holothuria VI 236–301
Egg
of cuckoo I 164–76, IV 363–73
relation between symmetry of egg and embryo in frog V 147–67, VII 148–209
of sparrow I 256
of tern X 144–68, XII 308–54
Egypt
Education in X 280–87
Egyptians
Anthropometry, Cairo-born natives XI 67–81
— of criminals XI 67–81
— of Modern VIII 66–78. Corrigenda, IX 332
Crania, observations on occipital bone (with plates) VIII 257–61
— pigmy, measurements and description of (with plates) VIII 262–66
cranial measurements, compared with Naqada I 425–65
— — — — — Negro VIII 306–20
— — — — — Moriori XI 95–133
— — — — — Tibetan XIV 208, 227–48
— — nasal characters IX 395–432
cranial types, compared with English and Negro VIII 135–38
— contours (plates) VIII 173–77
Head measurements XIII 17–24
prehistoric, see Naqada
Elderton, E. M.
Finger prints, inheritance of XIII 57–91
Infant Mortality and Employment of Women X 193–96
Inheritance of Intelligence, summary of present position with regard to XIV 378–408
Elderton, E. M. (*cont.*)
Life History Albums XII 373, 374
Pigmentation, stature and weight of schoolchildren VIII 340–53
Schoolchildren, association of drawing and other capacities in VII 222–26
Terns' eggs, co-operative study XII 308–54
Weight and Height of Glasgow Schoolchildren X 288–339
Women delinquents in New York, study of, review of XIII 305–8
Elderton, E. M. and B. M. Cave
Probable error of Biserial η, table to assist computation of XI 302
Elderton, E. M. and K. Pearson
Diphtheria attack and death-rate, influence of isolation on X 549–69
Hereditary character of general health IX 320–29
Natural selection in man X 488–506
Variate difference method XIV 281–310; see also X 488–506
Elderton, E. M. and E. H. J. Schuster
Inheritance of psychical characters V 460–69
Elderton, W. P.
Calculation and adjustment of moments and application of quadrature formulae IV 374–84
Goodness of fit, tables of P I 155–63
Interpolation, two independent variables II 105–8
— in n-dimensional space VI 94–103
Sickness-table, graduation and analysis of II 260–72. Addendum, II 503, 504
Tables of powers and sums of powers of natural numbers II 474–80
Elliot Smith, G.
Report on endocranial cast of Sir Thomas Browne's skull XV 67
Ellipse
Instrument for drawing III 469
Embleton, D. and T. Lewis
Split-hand and split-foot deformities, types, origin and transmission VI 26–58
— — — — (reference) VI 69–79
Emmons, A. B.
Variation in female pelvis IX 34–57
Employment of Women
and infant mortality X 193–96
English
Brain-weights and other anthropometrical characters IV 13–104, 105–23, 124–60
Crania, Moorfields, variation in and comparison with Whitechapel V 86–104
— Whitechapel, variation and correlation in, comparison with other races (with plates) III 191–244
Cranial measurements II 346–56
— — compared with Naqada I 424–65
— — — Negro VIII 302–24
— — — Moriori XI 95–133
— — — Tibetan XIV 208–49
— — nasal characters IX 395–432
— type contours, comparison of English with other races VIII 134–38

English (*cont.*)
Cranial type contours, comparison of Whitechapel with modern English VIII 131–33
— — — plates VIII 143–47, 191–201
criminals, anthropometry I 177–227, XI 1–37
— measurements of III 60–62
duration of life among I 50–89
Inheritance of physical characters II 357–462
Insane X 356–83
schoolchildren, intelligence and head measurements V 105–46
— pigmentation, disease, stature and weight (Birmingham) VIII 354–84
Enteric Fever
inoculation VI 366–75
Environment
and heredity, relative importance of X 183–87
influence of on aphis I 132, 133
— — conjugation in Paramecium V 213–97
— — development of egg in frog V 161–62, VII 149–207
— — Helix nemoralis I 468–92
— — mortality in man IX 58–68
— — Nassa obsoleta and Nassa trivittata II 24–43
— — poppy II 62–67, 83, IV 401, 402
— — termites VI 329–47
— — variation in Chilomonas V 53–72
Error
functional and mathematical, illustrated by the opsonic index VIII 203–24
Errors
law of, notices of papers on V 206–10
— see also Frequency Distributions
of observation, effect of on correlation coefficient VI 109, XII 357–59
— note on Burnside's paper XV 401–6
probable, see Probable Error
Erythrocyte counts
and altitude VIII 280–91
Eskimos
Cranial measurements IV 297, 298
— type contours compared with English VIII 135
— — — — Moriori XI 103–12
— — — — Tibetan XIV 222–49
— — — plates VIII 179–83
Euler (reference)
Euler-Maclaurin quadrature formula I 278
Eupagurus prideauxi (crab)
variation in II 191–210
Eustachius (reference)
sesamoids XIII 146
Everitt, P. F.
Quadrature coefficients, table XII 282–83
Tetrachoric Functions, tables VII 437–51
— — supplementary tables for high correlations VIII 385–95
Evolution I 321–44
cranial measurements, with ancestral tree VIII 302–22, 333–37
— — nasal bridge characters, with ancestral tree IX 401–36

Evolution (*cont.*)
Committee of Royal Society V 22–25
Darwinian theory VII 368–85
in man IV 233–85
— data for problem of I 30–49, 345–60
of sesamoids of the knee-joint (with plates) XIII 350–400
Excalation
in meristic series III 334–37
Expectation of Life I 50–89
change of, in 2000 years I 261–64
in Ancient Rome, Hispania, Lusitania and Africa IX 366–80
Exponential Binomial Limit
tables of X 25–35
Extrapolation
dangers of III 99–103
Eye-colour
and age in man III 462–66
hair-colour and, correlation between III 459–62
inheritance of in man II 237–40, III 149, 154, 167, V 471–76
intelligence and V 126–28, 130, 131, 133, 143, 144
pigmentation of human iris (with plates and pedigree) VIII 267–79
schoolchildren in Scotland VI 129–235. Supplement, 1–67
Scottish insane V 333–44
stature and weight, relation to VIII 340–53, 369–75
susceptibility to disease and VIII 13–39, 354–69

Facial Spasm
inheritance of (with pedigree) XIV 311–15
Fagus sylvatica (beech tree)
homotyposis and differentiation of leaves (with plates) III 104–7
Fallopius (reference)
sesamoids XIII 146
Fantham, H. B. (reference)
Trypanosome strains X 89–97
Fat, Butter, see Cows
Fawcett, C. D.
Crania, Naqada, variation and correlation of I 408–67
— (reference) III 191–244
frequency distribution for skull characters (reference) IV 37–40
Fay, E. A.
data on hereditary deafness IV 465–82
Fechner (reference)
Double Gaussian frequency distribution IV 196–99
Fecundity
in sows VI 433–36
See also Fertility
Femur
measurement of medieval English femora, reply to Dr Parsons X 573, 574
Fertility
among the deaf IV 474–78
and morphology of fruit of Sanguinaria VII 314–24

Fertility (*cont.*)
differential, effect of on degeneracy VII 258–75
in Aphis I 133–34
in Cercis canadensis, homotypic correlation of fertility characters XI 201–14
in Hibiscus, correlation with somatic characters VIII 52–65
in jellyfish I 255, 256
in man I 30–38
in man, and age at marriage, etc. XIV 337–54
— distribution of IV 247–56
— and duration of life IV 233–47
— and duration of marriage IV 264–80
— and occupation IV 280–85
in poppy II 66, IV 394–98
in rats, correlation with weight VII 381–85
in sows, VI 433–36. Inheritance of, V 203–5
inheritance of VII 368–85, VIII 244–47, 247–49
— in mice V 436–49
Ficaria (Celandine)
change in organic correlation during flowering season I 125–28
variation and correlation in II 145–64
variation in number of sepals and petals of I 11–20
Finger prints
association of X 421–78
inheritance of XIII 57–91
— note *re* collection of data II 356
Finns
head measurements IV 165, 295
Fir tree
variation in length of needles of I 21–23
Fish II 115–20, III 313–65, VIII 444–55
Fisher, R. A.
Small samples, frequency distribution of correlation coefficient in X 507–21
Fisk, E. L. (reference)
notice of paper on force of adverse selection among entrants at the extremes of life (insurance) VI 444
Fit, see Goodness of fit
Fitting of Curves I 265–303, II 1–23, XII 216–58
Fleming, A. (reference)
Opsonic Index VI 378–401, VIII 205–24
Flower, W. (reference)
crania III 192, 199, 200, 206
Flowering season
change in organic correlation during I 125–28
Foster-parents
of cuckoo, I 164–76; and size of eggs, IV 363–73
Fourfold Tables, see Tetrachoric
Fowl
inheritance of fecundity in poultry VII 370
leucosis in hen VII 234–36
malformation in head of IX 538–39
variation in comb of VI 420–32
Foxgloves (Digitalis gloxiniaeflora)
inheritance in XI 303–27, XIV 103–26
Foxgloves (*cont.*)
inheritance of abnormalities XIV 115–17
— coloration of corolla XI 312–17, XIV 108–14
— colour of spots and area of spotting XI 309–12, XIV 114, 115, 118–22
— pelorism XI 305–8, XIV 103–8
— ratio of width to length of corolla XI 318, 319, XIV 122–124
— seed length XIV 117
Fraternal Correlation, see Inheritance
French
cranial measurements II 347, 349, 355
— — compared with Naqada I 424–65
— — — Negro VIII 306–14, 321–24
— — — Tibetan XIV 210–26
— — — Whitechapel III 206–15, 221–26, 234–39
— — nasal characters IX 395
Frequencies of Occurrence
method of calculating from number of unequal probabilities VIII 413–19
See also VIII 396–403, 404–12, 430–36, IX 28–33
Frequency constants
"Best" values of XI 262–76
β_1 and β_2, high values of VIII 238
$\beta_6, \beta_5, \beta_4, \beta_3$ in terms of β_1 and β_2 VIII 146–47
Binomial, Probable Error of X 37–38
of a compound variable $[z=f(x_1, x_2)]$ deduced from constants of components VI 437–38
of index distributions, deduced from constants of components VII 531–41
of normal distribution determined from moments of truncated tail VI 63–68
Probable errors of, Pt I (1 variate), p.e. of moments, correlation between errors in moments II 273–81
Probable errors of Pt II (2 variates), correlation coefficient, product moments, correlation of errors in product moments, means of arrays, coefficients of regression line IX 1–10
Probable errors of Pt III, probable errors determined from grades XIII 113–32
— — Tables to facilitate computation of VII 128–47
Probable errors of β_1, β_2, p.e. of skewness, p.e. of distance from mean to mode VII 136–45
Value of higher β's in terms of β_1 and β_2 VII 146–7
Frequency distributions
compound, resolution of into components (Trypanosome strains) X 123–42
— in which components can be individually described by Binomial series XI 139–44
dimorphic, criteria for unimodality and bimodality III 84–98
— resolution of IV 230–31, XV 260–67
Edgeworth-Kapteyn IV 176–89, 199–204, V 168–71
Errors of type, diagram and table VII 386–97

Frequency distributions (*cont.*)
for anthropometric characters in Scottish Insane V 305–11
for Australian anthropometric data I 30–49
for brain-weights IV 37–42
for fecundity in sows VI 433–36
for haemacytometer counts V 351–60
for phagocytic counts VI 376–401; see also VII 210, 214
for sickness table II 264–72
Galton, McAlister and Fechner's Double Gaussian IV 176–89, 193–99
Goodness of fit, criteria of Edgeworth and Pearson compared XI 418–25
Mathematical expectation of Moments of XII 140–69, 185–210, XIII 283–95
Mean, Median and mode, relation of XI 425–29
moments, calculation of V 450–59
Normal IV 169–212
— curtailed IV 495–510, V 361–64
— determination of constants of, from moments of truncated tail VI 63–68
— deviations from V 470–76
of correlation coefficient in small samples IX 91–115, X 507–21, XI 328–413
of means of samples not drawn at random VII 210–14
of mean and standard deviation of arrays XI 239–61, XII 267–81
of moments of any order in samples, values of β-constants for XII 260–66
Pearson hypergeometrical IV 169–212, V 168–75
probability that two are samples from same population VIII 250–54, X 85–143
suitable for describing frequency of random sample V 172–75
type of, diagrams to determine from knowledge of β_1 and β_2 VII 131, 132
— probable error of, diagram and tables VII 386–97

Frequency ratios
classification of V 179–81

Frequency surfaces
bivariate, general forms when regression and variation are subjected to limiting conditions XV 77–88, 209–21
skew XIV 355–77, XV 222–30
non-skew XV 231–44

Fridolin, J. (reference)
crania, Moriori XIII 339

Frogs
symmetry of egg and embryo V 147–67, VII 148–209
See also Tadpoles

Fry, Agnes
Variation in leaves of mulberry I 258, 259

Fuegian
cranial measurements compared with Moriori XI 95, 98–102
— — — Tibetan XIV 222–26

Fuller, W. N. and R. Pearl
Variation and correlation in earthworm IV 213–29

Gaboon
crania, measurements of and comparison with other races (with plates) VIII 292–337
cranial measurements compared with Moriori XI 95–112, 125–32
— — nasal characters IX 394–437

Gaillardia aristata
Variation of flower heads in VI 106–8

Gain, E.
Pulmonaria officinalis III 398–458

Galen (reference)
sesamoids XIII 138–44

Galloway, A. R.
Canary breeding VII 1–42, VIII 435, 436
— — (reference) VII 398–400, 401–3
pigmentation of human iris VIII 267–79

Galton, F.
Ancestral Inheritance, Law of (reference) II 211–28
Biometry I 7–10
Difference problem I 385–99
Finger Prints (reference) X 422, 429–33, XIII 57–67
Frequency Distributions (reference) IV 193–99
Grades and Deviates V 400–6
History of Correlation (reference) XIII 33–41
Inheritance of Intelligence (reference) XIV 379, 408
Notice of lecture on "Probability, the Foundation of Eugenics" V 477
Organic correlation (reference) I 125–28

Gamma Function
On a formula for determining $\Gamma(x+1)$ VI 118, 119
Table of Log $\Gamma(p)$ VII 43–47
Incomplete, expansion in tetrachoric functions XIV 161–69, 175–80

Gauss, C. F. (reference)
Frequency Distribution IV 169–212
History of Correlation, Normal Curve, Method of Least Squares XIII 25–27, 30–32
Problem of inverse probabilities XIII 8–16

Gaussian Distribution, see Normal Curve

Gelasimus pugilator (crab)
asymmetry in II 307–20

German-Jews
hair- and eye-colour III 460

Germans
cranial measurements II 347, 355
— — compared with Naqada I 424–65
— — — Whitechapel III 206–15, 221–26

Gibson, W
Tables for facilitating computation of probable errors IV 385–93

Gilby, W. H.
Intelligence, Teachers' estimate of VIII 94–108
— — — (reference) XIV 384

Gillette (reference)
sesamoids XIII 155, 156

Giuffrida-Ruggeri, V. (reference)
Moriori crania, deformation of XIII 338–47

Gladstone, R. J.
Brain-weight and head size IV 105–23
— — — analysis of measurement IV 124–60
Glasgow
Schoolchildren, Pigmentation, stature and weight VIII 340–53
— Stature, weight and age X 288–339
Goodness of fit
Application of tables to test regression curves and theoretical frequency curves XI 239–61
application to Trypanosome strains X 85–143
"Best" values of frequency constants XI 262–76
of Frequency curves, criteria of Edgeworth and Pearson compared XI 418–25
of observation to theory in Mendelian experiments IX 309–14
Pearson's reply to critic, with special reference to fourfold tables XIV 186–91; further note on XIV 418
Probability that two independent distributions are samples from the same population VIII 250–54
Psychophysical curves XII 216–30
Tables of P for testing I 155–63
Theory of Multiple and Partial contingency XI 146–58
— application to differential death-rates XI 159–84
Gordon, Kate (reference)
Inheritance of Intelligence XII 367–72, XIV 388–94
Goring, C. B.
Criminological Work, reviews of VII 231–33
Obituary notice and appreciations XII 297–307
Goring, K. M.
Appreciation of C. B. Goring XII 301–7
Grades
probable error of frequency constants determined from XIII 113–32
Ranks and, Method of Correlation. Relation between actual variate correlation and correlation determined from ranks and grades X 416–18
— — — Correction for ties, experimental determination of probable error XIII 264–82
— — — application to examination results XIII 302–5
Grades and Deviates V 400–6
Graduation
of sickness table II 260–72. Addendum, II 503–4
— — by Makeham's hypothesis III 52–57
Greeks
anthropometry and cranial measurements XIII 92–112
education in Egypt X 280–87
Greenwood, M.
General Hospital population (reference) IV 19, 20
Goodness of fit, χ^2 test for fourfold tables (reference) XIV 189–91
Greenwood, M. (*cont.*)
Human viscera, weight variability and correlation III 63–83, IX 473–85
Sampling, errors of random (small samples) IX 69–90
Greenwood, M. and J. W. Brown
Weight of human viscera IX 473–85
Greenwood, M., L. D. Cripps and E. M. Newbold
Inter-relation of vital capacity, stature, weight, etc. XIV 316–36
Greenwood, M. and J. D. C. White
Opsonic Index VI 376–401, VII 505–30
— — (reference) VII 532–41, VIII 204–24
Greiner (reference)
Skew variation IV 169–212
Greyhound
inheritance of coat-colour in III 245–98
Grier, N. M.
Variation and distribution of leaves in sassafras XII 372, 373
Griffiths, G. B.
Head measurements of criminals III 60–62
Grouping
Correction for IX 316–20
See also Class-Index Correction, Sheppard's Corrections
Growth-rate
in trout VIII 449–55
Gruber, W. (reference)
sesamoids XIII 133, 137, 156–67, 350–56, 363, 367
Guanche
cranial type contours, compared with English VIII 134, 135
— — — — — Moriori XI 103–12
— — — — — Tibetan XIV 228–47
— — — plates VIII 185–89

Haemacytometer
error of counting with V 351–60
Hair-colour
and age in man III 462–66
and eye-colour, correlation between in man III 459–62
and intelligence V 126–32, 142
inheritance of III 149, 154, 168, V 471–76
of inhabitants of Eastern Mediterranean XIII 105–7
of schoolchildren in Scotland VI 129–235. Supplement, 1–67
of Scottish insane V 333–44
standard hair-scale (plate) V to face 474
stature and weight, relation to VIII 340–53, 369–75
susceptibility to disease and VIII 13–39
Hair-curliness
of schoolchildren III 154, V 126
Hair-set
correlation of intelligence and V 126–33, 143
inheritance of III 151, 154, 169
Hamy, E. T. (reference)
Crania XIII 338, 340, XIV 220

Hand
metacarpal bones I 350
phalanx, proximal and distal I 351–60
variability and correlation in I 345–60
Handwriting
inheritance of III 153–55, 190
and intelligence V 126–31, 133, 145
Hanel, E. (reference)
Pure Line, Hydra grisea VII 373–81
Hardy, G. F.
Notice of book of lectures on Construction of Mortality Tables VII 413
Harris, J. A.
Adoxa, variation in VII 218–22
Correlation between a variable and deviation of a dependent variable from its probable value VI 438–43
Correlation coefficient, calculation of for integral variates VII 214–18
— coefficients, inter- and intra-class IX 446–72
Fertility, correlation with somatic characters VIII 52–65
Homotyposis in Cercis canadensis XI 201–14
Intra-class correlation, spurious values X 412–16
Sanguinaria, morphology of fruit of VII 305–51
Staphylea, selection in VII 452–504
Weight of seed, influence of on plant IX 11–21, X 72–84
Hart, B. (reference)
Hierarchical order among correlation coefficients XII 356–66
Harvey, W. F. and H. W. Acton
Altitude and no. of erythrocytes VIII 280–91
Harvey, W. F. and A. G. McKendrick
Opsonic index VII 64–95
Haswell, W. A. (reference)
sesamoids XIII 389
Haviland, G. D. (reference)
Termites VI 329–47
Hawes (reference)
Anthropometry of inhabitants of Eastern Mediterranean XIII 95, 97, 103
Head measurements
and brain-weight IV 105–23, 124–60
and intelligence in schoolchildren V 119–25
inheritance of in schoolchildren III 137–45
in Man, comparison of sexes IV 161–68
— inter-racial correlation IV 286–312
of Afghans IV 293
of Ainos IV 165, 293
of Arabs IV 293
of Armenians IV 165, 293
of Buriats IV 293
of Chinese IV 294
of criminals in British Isles I 177–227, III 60–62
of Egyptians VIII 66–78, XI 67–81, XIII 17–24
of English IV 110, 111, 143
of Esthonians IV 295
of inhabitants of Eastern Mediterranean XIII 97–104, 107, 108
of Finns IV 165, 295
Head measurements (*cont.*)
of Jews IV 165, 293
of Kalmuks IV 165, 294
of Kurds IV 165, 294
of Letts IV 294
of Lithuanians IV 165, 294
of Livonians IV 294
of Oxford undergraduates VIII 47, 48
of Persians IV 295
of Poles IV 165, 295
of Russians IV 165, 293, 294, XV 254–70
— Great IV 293–94
— White IV 293–94
of Scottish insane V 300–33 and supplement
of Siberians IV 293–95
of Tatars IV 165, 295
of Turkomans IV 295
of Turks IV 295
of Uzbegs IV 295
See also Anthropometry
Health
General, Hereditary character of IX 320–29
— scale of IX 321
inheritance of in schoolchildren III 145–54
intelligence and V 126–32, 142
Hearing
Acuity of, and reaction times XV 346–72
Heart
weight and variability of in man III 63–83, IX 473–85
Height, see Stature
Heister, L. (reference)
sesamoids XIII 149, 150
Helguero, F. de
Dimorphic Frequency curves III 84–98
— — — resolution of IV 230–31
— — — — notice of paper on V 480
Variation and Homotyposis in Cichorium intybus V 184–89
Helix arbustorum
natural selection in V 387–99
Helix nemoralis
variation in I 468–92
Hen, see Fowl
Henderson, J.
Tetrachoric Functions, expansions in XIV 157–85
Variate Difference Method, Tables of Functions of Geometrical Decadence XIV 310
Henderson, R. (reference)
Frequency curves and moments, notice of paper VI 124
Henry, E. R. (reference)
Finger prints, classification X 421, 423, 447
Hensgen, C.
Variation in Helix nemoralis I 468–92
Heredity
and environment, relative importance of X 183–87
See also Inheritance
Heron, D.
Canaries, inheritance in VII 403–9
Correlation coefficient, partial, probable error of VII 411–12
— danger of some formulae VIII 109–22

Heron, D. (*cont.*)
Inheritance of insanity X 356–83
Probable error, abac for determining p.e. of correlation coefficient VII 410, 411
Reproductive Selection X 419–20
Sex-ratio, inheritance of V 79–85, VI 120–22
Heron, D. and K. Pearson
On Theories of Association IX 159–316
Hervieux (reference)
Evolution of canary VII 11–18
Hessians
Brain-weights and other anthropometric characters IV 13–104, 133
Heterogeneity
Coefficient of class heterogeneity V 198–203
of material, compound frequency distributions in which components can be described by Binomial series XI 139–44
Trypanosome Strains X 119–42
See also under Homogeneity
Heterostylism
in Pulmonaria officinalis III 398–458
Heyer, A.
Variation in length of needles of conifers VI 354–65
Heymans, G.
Data on inheritance of psychical characters V 460–69
Hibiscus
Correlation of Fertility and somatic characters VIII 52–65
Intra-class correlations IX 451–54
Hierarchical
order among correlation coefficients XII 355–66, XV 150–60
Hildebrand (reference)
Pulmonaria officinalis III 400, 401, 423, 424, 449
Hindus
cranial measurements, compared with Burmese XIII 235–47, 250, 251
— — — Tibetan XIV 206–26
— — nasal characters IX 395, 399, 400, 404–21, 425, 427, 429, 432
Hirshberg, L. K.
Selection and Intermediates in Bacillus coli IX 331, 332
Holothuria
Variation, development and growth in VI 236–301 (with plates)
Homogamy, see Assortative-mating
Homogeneity
of material, brain-weights IV 29–36
and Heterogeneity in Collections of Crania II 345–47, 504–8
— — — Naqada Crania I 422–24
— — — Whitechapel Crania III 206
and Heterogeneity of material, Scottish insane V 315–38
Homogyne alpina
variation in I 24–26
Homotypic correlation IX 456–58, 463, 464
Homotyposis I 321–44
in Beech leaves III 104–7
in blood corpuscles of tadpoles VI 403–19
Homotyposis (*cont.*)
in Cercis canadensis, fertility characters XI 201–14
in Cichorium intybus V 187–89
in mulberry leaves I 258
in Paramecium caudatum I 400–7
in Shirley poppy II 67–71, IV 409–10
in terns' eggs X 144–68, XII 331–35, XV 324–9
Honduras
note on piebald boy (with plates) IX 330, 331
Horses
American trotters II 2–6
inheritance of coat-colour I 361–64, II 229–34
Sex-ratio, V 83–85. Inheritance of, VI 120–22
Hosny, M.
Education in Egypt X 280–87
Hospital Population IX 473–85
compared with general population III 65, IV 127–32
Houses
distribution of cases of disease in VIII 396–403, 404–12, IX 28–33
Cancer-houses VIII 430–36; see also VIII 413–19
Howitt, Mary (reference)
connection with Weldon V 4, 5
Howitt, William (reference)
connection with Weldon V 3, 4
Humphrey, G. (reference)
sesamoids XIII 153, 355
Hungarians
cranial measurements II 338–49
Hurst, C. C. (reference)
Pigmentation of Iris VIII 267–79
Hutchinson, J. (reference)
Inter-relations of vital capacity stature and weight XIV 316–23, 333, 336
Huxley, T. H. (reference)
connection with Weldon V 2, 20–21, 29
on Origin of Species V 272
Pearson, Huxley Lecture III 131, 147
Hyalopterus trirhodus (aphis)
Variation and parthenogenetic inheritance in I 129–54
Hydra grisea
Pure line VII 373–81
Hyrtt, J. (reference)
sesamoids XIII 154, 171

Identification of Criminals I 205–13
Illegitimate Children
death-rate in first year of life IV 511
Immunity
smallpox and vaccination IV 313–31, 483–504, 504–10
statistical study in V 423–34
Independence
Theory of complete, of series of attributes II 127–32
Index distributions
deduction of constants VII 531–41
Inequalities
with possible applications in the theory of probability (limits to tails of frequency distributions) XV 245–53, 421–3

Infant mortality IV 277–80, 510–16
and education in Egypt X 284–87
and employment of Women X 193–96
and mortality of childhood X 488–506, XIV 300–8
Inheritance
Ancestral law of, see Ancestral Inheritance, law of
Bateson's revision of Mendel's theory II 286–98
Determinantal theory of V 44–45
— mathematical theory from notes and suggestions of Weldon VI 80–93
in bean II 499–503
in canaries VII 1–42, 398–403, 403–9, VIII 435–36
in cows, production of butter fat VII 106–26
— quantity and quality of milk VII 548–50
in foxgloves XI 303–27, XIV 103–26
in Man, note *re* Treasury of Human Inheritance VI 327, 328
in peas, Mendel's law I 228–54
in poppies II 56–100, IV 394–426
in statoblasts of Pectinatella magnifica I 128
influence of change of sex on intensity of II 237–40
Mendelian, see Mendelism
of brachydactyly VI 327
of coat-colour in cattle IV 427–64
— — greyhound III 245–98
— — horse I 361–64, II 229–36
— — mice II 101–4, 165–73, 282–85, 294–98, III 1–51, IV 1–12
— — rabbits II 299–306
of deafness IV 465–82
of duration of life I 50
of eye-colour VIII 267–79
of facial spasm (with pedigree) XIV 311–15
of fertility VII 368–85, VIII 244–47, 247–49
— in mice V 436–49
of general health IX 320–29
of hairiness in Lychnis II 47–55
of insanity X 356–83
of intelligence XII 367–72, XIV 378–408
of meristic characters in Spinax niger III 341–44, 363–65
of physical characters in man, stature, span and forearm II 357–462
of psychical characters in man III 131–90, V 460–69
of sex-ratio VI 109–11
— — in horses V 83–85, VI 120–22
— — in man V 73–78, V 79–85
— — in mice V 436–49
of size of litter in mice V 436–49
— — — sows V 203–5
of skin-colour, crosses between Negro and White VI 348–53
— — piebald family (with plates and pedigree) X 197–200
of split-foot and split-hand deformities (with pedigree and plates) VI 26–58, 117, 118
— — and lobster claw (with pedigree and plates) VI 69–79, IX 381–90
Addendum IX 540
Inheritance (*cont.*)
of taste and occupation in man III 467–69
parthenogenetic, in aphis I 129–54
Piebald family X 197–200
racial, and Mendelism VIII 424–30
stability of population, conditions for X 175–78
standard deviations of fraternal and parental correlation coefficients XIV 1–22
White-lock family (with pedigree and plates) XIII 347–49
Inion
difficulties of determination of III 375–77, 391, XIII 184, 185
Inoculation
anti-typhoid, statistical study of VI 366–75
Insane
Anthropometry of Scottish V 298–350 and supplement
Insanity
and cancer VII 290–95
inheritance factor in X 356–83
Insectivores
sesamoids in XIII 356
Integration
finite difference formulae for I 272–303
Intelligence
and age, Drinkwater's data XIV 396–98
— Pearson's data V 112–19
and brain-weight IV 78–80, 83
and cranial capacity III 392–96
and head measurements and other physical and psychical characters V 105–46
and home conditions in schoolchildren (critical notice) XV 161–72
and temperature, pulse and breathing in criminals XI 1–19
inheritance of, comparison of series tested by Binet-Simon method and series measured by broad categories XII 367–72
— in schoolchildren III 147–49
— summary of present position with regard to XIV 378–408
Mathematical VII 352–67
Quotient, correlation of with Teachers' estimate XIV 396, 397
Scale of V 106–12
Teachers' estimate VII 542–48, VIII 79–93
— — significance of VIII 94–108
Inter- and Intra-class correlation IX 446–72
Intercalation
in meristic series III 337, 338
Interpolation
by finite differences, two independent variables II 105–8
in n-dimensional space VI 94–103
review of paper, by Narumi, on XIV 420
Inter-racial Correlation I 460, IV 162, 163, 286–312, IX 423–33, XIV 193, 194
Coefficient of Racial likeness XIII 247–51, XIV 205–27
Cranial and head measurements II 347–56
See also Inter- and Intra-class correlation

Inter-racial Variation
coefficient of IV 290
Intra-class Correlation IX 446–72, X 412–16
Introspection, see Self-consciousness
Irwin, J. O.
Terns' eggs, co-operative study XV 294–345
Isaacs, E.
Terns' eggs, co-operative study XII 308–54
Isolation
and Diphtheria attack and death-rate X 549–69
Tuberculosis and segregation X 530–48
Isotropy
of frequency distributions V 470–76
Isserlis, L.
Corrigenda for paper on real significance of Probable Errors of frequency distributions, values of β-constants for distributions of moment coefficients of any order in samples XII 261–66
Formula for determination of cranial capacity from external measurements X 188–93
Mean values of products of deviations of mixed moment coefficients XII 183–84
Partial correlation Ratio, Pt I Theoretical X 391–411. Corrigenda, X vi
— — — Pt II Numerical XI 50–66
Probable errors and correlation coefficient of multiple frequency distributions with skew regression XI 185–90
Product-moment coefficients, high-order XII 134–39. Corrigenda, XII 266
Representation of statistical data, frequency curves and goodness of fit criterion XI 418–25
Schoolchildren, intelligence and home conditions (reference) XV 161–72
Italians
Correlation between hair- and eye-colour III 460, 461
in Egypt X 280–87
Italy
Economic Index Values, Mortara's Table. Correlation by Variate Difference Method X 340–55

Jacob, S., A. Lee and K. Pearson
Inter-racial correlation, cranial and head measurements II 347–56
Javanese
cranial measurements, nasal characters IX 395, 399–432
Jellyfish (Aurelia aurita)
variation in I 90–108
variability, symmetry and fertility in I 255
Jenkinson, J. W.
Frogs, symmetry of egg and embryo V 147–67, VII 148–209
Trout, growth variability and correlation in VIII 444–55
Jennings, H. S. (reference)
Pure line, Paramecium VII 370–72, 380, 381
Jews
hair- and eye-colour III 460
head measurements IV 165, 293
Jews (*cont.*)
measurements of XIII 108
and Gentiles, Mendelism and racial heredity VIII 424–30
Johannsen, F. (reference)
Inheritance in Phaseolus vulgaris II 499–503
Johannsen, W.
Notice of paper on cephalic index V 482
John Dory
number and arrangement of bony plates (with plates) II 115–20
Jones, H. G.
Intelligence, Teachers' opinion VII 542–48
Judgments
correlation of successive XIV 23–102

Kaffir
crania, measurements and comparison with other races VIII 292–337
Kalmuks
Cranial measurements IV 297
head measurements IV 165, 294
Kapteyn, J. C. (reference)
Frequency distributions IV 199–204, V 168–71
Karn, M. N.
Terns' eggs, co-operative study XV 294–345
Keith, Arthur
Note on Sir Thomas Browne's skull XV 1–2
sesamoids (reference) XIII 361–63
Kellogg, V. L. (reference)
Variability in bees V 420
Kidneys
weight of in man III 63–83, IX 473–85
Koga, Y. and G. M. Morant
Degree of association between reaction times for different senses XV 346–72
Koopmans, A. N. and Cool, C.
Umbelliferae, variation and correlation of number of umbel rays XI 38–49
Kribs, H. G.
Carabus auratus, sex variability in VI 103–5
Kurds
head measurements IV 165, 294
Kurtosis
of Frequency curves IV 172–212

Laplace (references)
History of Correlation, Normal curve XIII 25
Normal Curve IV 189–93
Problem of inverse probabilities XIII 1–16
Laski, H. J.
Mendelism and racial heredity VIII 424–30
Latter, O. H.
Cuckoo's eggs I 164–76, IV 363–73
Wasps X 201–7
Laxton, T. (reference)
Inheritance in peas I 237, 245, 251
Least Squares
Impossibility of, for systematic fitting I 295
Method of, for curve-fitting I 279–82, II 9–12
Lee, Alice
Capacity of skull, reconstruction of from external measurements (reference) III 366–97

Lee, Alice (*cont.*)
Gaussian Tail Functions X 208–14
inheritance in racehorses II 234–36
Tetrachoric groupings, further supplementary tables for determining high correlations XI 284–91
Tuberculosis and segregation X 530–48
Variation and correlation in plants I 316–19
Lee, A., A. Barrington and K. Pearson
Inheritance of coat-colour in greyhound III 245–98
Lee, A., S. Jacob and K. Pearson
Inter-racial correlation, cranial and head measurements II 347–56
Lee, A. and K. Pearson
Inheritance of physical characters in man II 357–462
Probable error in multiple normal correlation VI 59–68
Lee, A., B. M. Cave, H. E. Soper, A. W. Young and K. Pearson
Small samples, distribution of correlation coefficient in XI 328–413
Lee, A., A. Wright and K. Pearson
Wasps V 407–22
Length of life, see Duration of life
Lenz, F.
Attempt to ascertain the prevalence of syphilis in a large urban population (reference) VIII 437
Lepidium draba
variation in flowers of X 215–68
Lepidoptera
variation and elimination in III 113–30
Leucoderma
Nyasaland natives IX 353–58
Leucosis
in a hen VII 234–36
Lewenz, M. A.
Comparison of results for hand measurements and finger prints (reference) X 451
Lewenz, M. A. and K. Pearson
Measurement of capacity from cranial circumferences III 366–97
Lewenz, M. A. and M. A. Whiteley
Variability and correlation of human hand I 345–60
Lewis, T. and D. Embleton
Split-hand and split-foot deformities VI 26–58. Addendum, VI 117, 118
Note on inheritance of brachydactyly VI 327
(reference) VI 69–79
Life-History Albums XII 373, 374
Life-tables
Makeham's curve I 298–303
Limit
to probability of deviations of given sizes without assumption as to distributions, Generalised Tchebycheff Limits XII 284–96
Lindsay, D. E. (reference)
Dietaries IX 530–33, X 169–74
Linearity
Test for linearity of regression, Blakeman's criterion IV 332–50
Lithuanians
head measurements IV 165, 294
Litter
inheritance of size of, in mice V 436–49
— — in sows V 203–5
Liver
weight of in man III 63–83, IX 473–85
Lloyd, R. E. (reference)
Fertility in rats VII 381–85, VIII 244–47, 247–49
Lobster-claw, see Split-foot
Lock, R. H. (reference)
notice of book on Variation, Heredity and Evolution V 478–80
Long Barrow
Crania, measurements and descriptive catalogue IV 351–62
Cranial measurements, compared with Whitechapel III 207, 243, 244
— — — Moorfields and Whitechapel V 104
— — — Moriori XI 95–102
Lucas, E. V.
Appreciation of Goring XII 300, 301
Ludwig, F.
Variation problems in plants I 11–29
— — — (reference) I 316–19
Lumbricus (earthworm)
Variation and correlation in IV 213–29
Lung capacity
relation to other physical characters VIII 44–46, 49–51
See also Vital capacity
Lungwort (Pulmonaria officinalis)
Variation in flowers and heterostylism III 398–458
Luschan, F. von (reference)
Anthropometry, inhabitants of Eastern Mediterranean XIII 92, 98, 102–7
Lutz, F. E.
Change of sex and inheritance II 237–40
Lychnis diurna and Lychnis vespertina
Inheritance of hairiness in II 47–55

Macalister, A. (reference)
sesamoids XIII 154, 155, 171, 361, 382
McAlister, D. (reference)
frequency distributions IV 193–99
MacCormick, A. (reference)
sesamoids XIII 382
Macdonald, D.
Pigmentation and susceptibility to disease VIII 13–39
— — — (reference) VIII 368, 369
Macdonell, W. R.
Crania (references) VIII 308, 309, 312, XI 101, 106, XIII 178–80
— English, Moorfields V 86–104
— — Whitechapel III 191–244
Cranial measurements, Naqada (reference) I 429
Criminal anthropometry I 177–227
Expectation of Life in Ancient Rome IX 366–80
Frequency distribution for skull characters (erence) IV 37–40

Macdonell, W. R. (*cont.*)
Obituary Notice XI 281–83
Smallpox and Vaccination I 375–83, II 135–44
McDougall
spot test (reference) VIII 41
McIntosh, D. C.
Variation in Ophiocoma nigra II 463–73
McIntosh, J. and J. E. Adler
Histological Examination of case of Albinism VII 237–43, (reference) VII 244, 245
Maciver, D. R.
Cranial measurements I 424–27
McKendrick, A. G.
Frequencies of occurrence calculated from number of unequal probabilities VIII 413–19
McKendrick, A. G. and W. F. Harvey
Opsonic index VII 64–95
Mackenzie, W. C. (reference)
sesamoids XIII 384, 385
Maclaurin (reference)
Euler-Maclaurin quadrature formula I 273
Expansion and curve-fitting I 267–72
MacLeod, J. (reference)
Organic correlation in Ficaria ranunculoides I 125–28
McMahon, J.
hyperspherical goniometry and its application to multiple correlation XV 173–208
McMullan, G. and K. Pearson
Split-foot, inheritance of IX 381–90
Maize
and De Vries' mutation theory I 369
Makeham (reference)
Fitting of curve to mortality statistics I 298–303
Hypothesis of, applied to graduation of sickness table II 503, 504, III 52–57
Malaria
and Mosquitos VII 421–36
Malays
Cranial measurements, compared with Burmese XIII 235–47, 250, 251
— — — Tibetan XIV 206–26
— — nasal characters IX 395, 400–7, 410–21, 425, 427, 432
Manners-Smith, T. (reference)
sesamoids XIII 377, 383
Mantis religiosa
colour in III 58–59
Maori
Cranial measurements, compared with Moriori XI 95–102
— — — Tibetan XIV 222–26
— — nasal characters IX 395
Maravar (Dravidians)
Cranial measurements, compared with Burmese XIII 238, 239, 244–47, 250
— — — Tibetan XIV 206–26
Marchand (reference)
brain-weight IV 16
Marriage
ante-nuptial conceptions and births in Australia I 31
Marriage, age at II 20–23
and duration of life IV 256–63
— — marriage and number of offspring IV 264–80
and fertility XIV 337–54
Marshall, John (reference) IV 16
Marsupials
sesamoids in XIII 371–75
Martin, R. (reference)
Crania XI 94, 99, 100
Mathematics
Intelligence in VII 352–67
Matiegka, H. (reference)
Brain-weight IV 17, 18
Maynard, G. D.
Anti-typhoid inoculation VI 366–75
Cancer death-rates VII 276–304
Diabetes death-rates VIII 225–31
Human Fertility, study in XIV 337–54
Piebald Negro X 193
Shirley poppies VII 227–30
Maynard, G. D. and J. McD. Troup
Distribution of cases of disease in houses VIII 396–403
Mazes
Association of steadiness of hand with rapidity and artistic capacity XII 170–77
Mean
Means of samples not drawn at random, distribution of VII 210–18
Median and Mode, relation of in Frequency curves XI 425–29
Probable error of distance from Mode to VII 129, 130, 142, 143
von Török's attack on II 339–45, 508–12
See also Frequency constants, Frequency distributions, Probable error and Small samples
Measles and Pigmentation VIII 25–39, 360–65
Meckel, J. F. (reference)
sesamoids XIII 351–57, 362, 363, 368–71
Mediterranean
anthropometry of inhabitants of Eastern XIII 92–112
Mendelism V 36, 39, 42, 47, 48
Albinism in Sicily III 107, 108. Note on, III 471, 472
Alternative Inheritance, law of in peas I 228–54
Ambiguity of Mendel's categories II 44–55
Ancestral Inheritance, law of, and II 211–29, III 109–12, VIII 241–243
Bateson's revision of theory II 286–98
Canaries VII 1–42, 403–9
Coat-colour in cattle IV 427–64
— — greyhound III 248, 263
— — mice II 101–4, 165–73, 282–85, III 1–51, IV 1–12
— — rabbits II 299–306
Correlation Coefficient and VIII 420–24
Eye-colour VIII 267–79
Fowl, variation in comb of VI 420–32
Foxgloves, inheritance correlations in XI 323–27, XIV 103–26
Goodness of fit of observation to theory in Mendelian experiments IX 309–14

Mendelism (*cont.*)
Linearity of regression and VIII 421–23
Mathematical Theory of Inheritance and VI 80–93
Mendelian and anatomical units in connection with nasal bridge of crania IX 423, 424, 434
Mice-breeding II 101–4, 165–73, 282–85, III 1–51, IV 1–12
Piebaldism IX 330
Probable error of Mendelian class frequency XI 429–32
Rabbit-breeding II 299–306
Racial heredity and, Jews and Gentiles VIII 424–30
Skin-colour crosses between Negro and White VI 348–53
Split-foot and Lobster-claw VI 69–79, IX 383–90
Split-foot and split-hand VI 50–56, 117, 118
Sex-ratio, inheritance of and V 77, 79
Spinax niger III 345, 346, 363–65
Variation and VIII 423
Yule's coefficient of Association and Colligation and IX 206–13
Mental and Moral Characters, see Psychical Characters
Mental Defect
and pigmentation VI 197
in criminals XI 7–15
Mérejkowsky, C. de (reference)
Crania, nasal measurements VIII 316, 317, 320, IX 392–95, 398, 405
Merism
and sex in Spinax niger III 313–65
Mice V 41, 44
Breeding experiments, records of matings (Weldon) XI. Appendix 1–60
Inheritance of coat-colour in II 101–4, 165–73, 282–85, 294–98, III 1–51, IV 1–12
Inheritance of sex-ratio and size of litter V 436–49
Milk
inheritance in production of butter fat VII 106–26
— of quantity and quality of VII 548–50
Mode
Criteria for uni- or bi-modality of dimorphic frequency curves III 84–98
Determination of I 260, 261
Median and Mean, relation of in Frequency curves XI 425–29
Probable Error of distance from mean to VII 129, 130, 142, 143
Mollusca I 109–24, 468–92, II 24–43, III 299–306, V 32–34, 37, 38, 387–99
Molluscoidea I 128
Moment Coefficients
Corrections for, when there are finite or infinite ordinates and any slopes at terminals of range of frequency distribution, Non-Asymptotic Curves XII 231–51
— — — — Asymptotic Curves XII 251–58
Deviations of, mean values of products of XII 183–84
Moment Coefficients (*cont.*)
Values of constants β_1 and β_2 for distributions of moment coefficients of any order in samples XII 260–66
Variation of in samples XII 140–69, 185–210, XIII 283–95
See also Moments
Moment Functions
Incomplete Normal, with tables VI 59–68
Moments
Adjustment of IV 380–84
Calculation of crude IV 374–78
Corrections for, Sheppard, calculation of V 450–59
— — Elementary proof of III 308–12
Curve fitting by method of I 265–303, II 12–23
determination of those of whole distribution from those of quadrant XIII 424–25
Incomplete Moments of Normal Solid XIII 401–25
Mathematical expectation of moments of frequency distributions XII 140–69, 185–210, XIII 283–95
of trapezoidal areas II 7
Probable errors of II 273–81
Sheppard's corrections for III 308–12, V 450–59
See also Frequency constants, Moment coefficients, Product moments
Monotremes
sesamoids in XIII 383–86
Moorfields Crania
Variation and correlation in, and comparison with Whitechapel (with plates) V 86–104
Comparison with Negro VIII 302–24
— — Tibetan XIV 208
— — Whitechapel III 212, 243, V 86–104
Morant, G.
Random occurrences in space and time XIII 309–37
Tibetan Crania XIV 193–260
Morant, G. M. and Y. Koga
Degree of association between reaction times for different senses XV 346–72
Morgagni, J. B. (reference)
sesamoids XIII 151
Moriori
Crania, measurements, contours and comparison with other races (with plates) XI 87–135
Cranial measurements, compared with Tibetan XIV 210–49
Question of artificial deformation of (with plates) XIII 338–47
Morphology
of fruit of Sanguinaria VII 305–51
Mortality
Gompertz Law of II 261
Social distribution of, from different causes in England and Wales, 1910–12 XV 382–400
Mortara, G. (reference)
Economic Index values for Italy X 340–55
Moschatel (Adoxa Moschatellina)
Variation in II 108–13, VII 218–22

Mosquitoes
and Malaria VII 421–36
Moths V 32
Mott, F. W. (reference)
Anticipation and Insanity X 356–83
Mouse, see Mice
Mulberry
Variation in leaves of I 258, 259
Müller, G. E. (reference)
Curve-fitting, psychophysical data XII 220–27
Multiple Contingency, see Contingency
Multiple Correlation, see Correlation, multiple
Mumford, A. A. and M. Young
Inter-relationships of physical measurements and vital capacity XV 109–33
Mutation
De Vries' Theory I 365–74
Myers, C. S. (references)
Anthropometrical statistics, Egyptian VIII 67–68
Homogeneity in collections of crania II 345–47, 504–8
Myres, J. L. (reference)
Inhabitants of the Eastern Mediterranean XIII 94

Naqada
Crania, variation and correlation in, comparison with other races (with plates) I 408–67
— See also On homogeneity in collections of crania II 345–47, 343–56
Cranial measurements compared with English, Whitechapel III 207–15, 221–26, 232–39
— — — Negro VII 302–14, 321–24
— — — Tibetan XIV 208–26
Narumi, S.
Bivariate frequency distributions, general forms when regression and variation are subjected to limiting conditions XV 77–88, 209–21
Further inequalities with possible applications in the theory of probability (upper and lower limits to tails of frequency distributions) XV 245–53
— — note on XV 421–23
Review of paper on interpolation by XIV 420
Nassa obsoleta and Nassa trivittata (snails) II 24–43
Natural Selection, see Selection
Neanderthal Skull II 352, 353
Negro
Crania, Congo-Gaboon, Measurements and comparison with other races (with plates) VIII 292–337
— Formulae for determination of capacity from external measurements X 188–93
Cranial measurements, compared with Moriori XI 95–112, 125–32
— — — Naqada I 426–65
— — — Tibetan XIV 208–47
— — Nasal characters IX 394–437
— Type, compared with English VIII 134–38
— — Contour Plates VIII 155–71
Negro (*cont.*)
Piebald X 193
Skin-colour of crosses between Negro and White VI 348–53
Nesbit, R. (reference)
sesamoids XIII 136, 137
Nettleship, E. (reference)
night-blindness VI 71, 73, 118
Newbold, E. M.
Note on paper by Burnside on errors of observation XV 401–6
Newbold, E. M., L. D. Cripps and M. Greenwood
Inter-relation of vital capacity, stature, weight, etc. XIV 316–36
Newsholme, A. (reference)
Tuberculosis X 530–48
Newton (reference)
Quadrature formula I 275
New Zealand data
Study in Fertility XIV 337–54
Nichols, J. B. (reference)
Numerical proportions of Sexes at birth. Notice of book VI 123
Night-blindness
Nettleship family VI 71–73
Noise, see Vivacity
Noorduyn, C. L. W. (reference)
Canary-breeding VIII 435–36
Normal Correlation
Incomplete moment functions VI 59–68
Surface, 2 variables, product moments XII 86–92
— n variables, product moments XII 134–39, 177–83
Normal Curve
Fitting of, when part only of frequency is given II 1–7
Instrument for drawing III 469
Probable error of skewness and modal divergence, calculation of V 190
Problem of Inverse probabilities XIII 8–16
Sheppard's Tables of areas and ordinates II 174–90
Table of deviates for each permille of Frequency V 405
— ordinates for each permille of Frequency XIII 426–28
— "Tail" functions VI 68
— — — when tail is more than half the curve X 208–14
Normal Solid
Incomplete moments XIII 401–25
Nubians
cranial measurements, nasal characters IX 395, 403–7, 421, 425, 427, 429, 432
Nucleus
Mass relation of nucleus and cytoplasm II 241–54
Nyasaland Natives
Congenital anomalies (with plates) X 1–24
Pigmentation anomalies (with plates) IX 333–65
Nystrom (reference)
Hypothesis as to head-form IV 286

Observations
Choice in distribution of XII 1–85
Obstetric Prediction IX 486–529
Abac for determination of Conjugate IX 539–40
Occipital Bone
Observations on, Egyptian Crania VIII 257–61
Occipital Index
Tables for calculating XIII 261, 262
Occupation
and cancer and diabetes VII 288, 289
Contingency between occupations of father and sons III 467–69
and number of offspring IV 280–85
and smallpox vaccination II 138–44
Offspring
number of, correlation with duration of life IV 233–47
— — duration of marriage IV 264–80
Ophiocoma nigra (Brittle-star)
Variation in II 463–73
Opsonic Index
Mathematical and Functional Error VIII 203–24
Phagocytic Counts, frequency distributions of VI 376–401
— — distribution of means of samples VII 505–30
— — frequency constants VII 531–41
Reliability of for diagnosis, etc. VII 64–78
Comparison of indices calculated by different observers VII 79–95
Oraons of Chota Nagpur
head measurements II 346, 352–56
Order
Effect on correlation of changing order in scale V 176–78
Orensteen, M. M.
Anthropometry, Cairo natives XI 67–81
Head measurements, Egyptian Natives XIII 17–24
Orschansky, O. (reference)
Inheritance of sex in man V 72, 78
Ottomans
in Egypt X 280–87
Owen, R. (reference)
sesamoids XIII 351, 356, 358, 365, 368–71, 377
Oxford Undergraduates
Acuity of Vision VIII 43
Head measurements VIII 47, 48
Physical development VIII 44–47

Pairman, E. and K. Pearson
Moment coefficients, corrections for XII 231–58
Parabola
Instrument for drawing III 469
Paramecium I 400–7
Conjugation in V 213–97, VII 372
Variation in V 53–72
Parental Correlation, see Inheritance
Parisians
Cranial measurements compared with Whitechapel III 221–26
Parker, K. M., W. Rowan and J. Bell
Terns' eggs X 144–68
Parker, T. J. (reference)
sesamoids XIII 389
Parkes, A. S.
Sex-ratio and related phenomena in pig-litters XV 373–81
Parsnip XI 39–49
Parsons, F. G. (reference)
Crania XIV 221, 222
English Femur X 209–13, 573, 574
sesamoids XIII 355, 357, 359, 368, 369, 372, 374, 381
Parthenogenetic Inheritance I 129–54, 400
and Pure Line Theory VII 372–81
Partial Contingency, see Contingency
Partial Correlation, see Correlation, partial
Pastinaca sativa XI 39–43
Paton, D. N.
Dietaries, Statistical study of X 169–72
Pauperism
and Tuberculosis X 530–48
Pea
Inheritance in, Mendel's Law I 228–54, II 45
Peacock (reference)
Brain-weights III 63, 68, IV 31, 129, 133
Pearl, M. D. and R. Pearl
Variation in comb of Domestic Fowl VI 420–32
Pearl, R.
Brain-weight, variation and correlation in IV 13–104
— — comparison with Gladstone's material IV 132–44
— — (reference) IX 478–80
Chilomonas, variation in, under favourable and unfavourable conditions V 53–72
Duration of Life, mean, for those dying within the first year of life IV 510–16
Fertility in Poultry (reference) VII 370, 371
Frequency constants of compound variable $z = f(x_1, x_2)$ VI 437, 438
Mendelism (reference) IX 310–14
Paramecium, conjugation in V 213–97
Probable error of Mendelian Class Frequency, criticism of (reference) XI 429–32
— errors of skewness and modal divergence of Normal Curve V 190
— — of standard deviation, points concerning VI 112–17
Pearl, R. and F. J. Dunbar
Variation and correlation in Arcella II 321–37
Pearl, R. and W. N. Fuller
Variation and correlation in earthworm IV 213–29
Pearl, R. and M. D. Pearl
Variation in comb of Domestic Fowl VI 420–32
Pearson, E. S.
Natural Selection and the Age and Area theory of Dr Willis XV 89–108
Personal Equation, Variations in and correlation of successive judgments XIV 23–102
Probable Error of Class-Index Correction XIV 261–80

Pearson, E. S. and Karl Pearson
Polychoric coefficients of correlation XIV 127–56
Pearson, H. S.
Terns' eggs, co-operative study XV 294–345
Pearson, Karl
Albinism, internal VII 244–47
Ancestral Heredity, Law of II 211–29
— Inheritance VIII 239–43
Association, surface of constant IX 534–37
Biology, Fundamental conceptions of I 320–44
Broad Categories, influence of on Correlation IX 116–39
Brain-weights (reference) IV 15, 29, 47
Cancer Houses VIII 430–35, IX 31–33
Compound Frequency Distributions in which components can be described by Binomial series XI 139–44
Correction of misstatement by Greenwood on Phthisis death-rates X 418, 419
Correlation, effect of changing scale order on V 176–78
Correlation coefficient, bi-serial expression for VII 96–105
— — Partial and Multiple, properties of, and relations between coefficients of different orders XI 231–38
— Ratio, Bi-serial and tri-serial VII 248–57
— — Correction for, comparison with mean value for zero correlation VIII 254–56, XIV 412–17
Cows, inheritance in VII 548–50
Crania (reference) III 207, 221
Cranial measurements (reference) I 409–13, 458, 461
— Types II 509–12
Curve fitting I 265–303, II 1–23
Curves for describing Frequency of Random samples V 172–75
Darwinism, Biometry and some recent Biology VII 368–85
Determination of Mode I 260, 261
Disease, multiple cases of in Same House VIII 404–12, IX 28–33; see also VIII 430–36
Dietaries IX 530–33, X 172–74, (reference) X 169–72
Expectation of Life, change of in 2000 years I 261–64
Femur measurements, reply to Dr Parsons X 573, 574
Fertility and Degeneracy VII 258–75
— Inheritance of VII 381–85, VIII 244–47, 247–49
Frequency constants of index distributions VII 531–41
— surfaces, skew XV 222–30
— — non-skew XV 231–44
Fundamental Problem of Practical Statistics. Bayes' Theorem XIII 1–16. Note, XIII 300–1
Galton's Difference Problem I 385–99
Goodness of Fit in Statistics and Physics XI 239–61; see also XII 267–81
Pearson, Karl (*cont.*)
Goodness of Fit. χ^2 test XIV 186–91
— — note on XIV 418
Hair- and Eye-colour, correlation between in man III 459–62
Heredity in Sex, remarks on Cobb's note VI 109
Heterogeneity, coefficient of class V 198–203
History of Correlation XIII 25–45
Homogeneity and Heterogeneity in Crania II 345–47
Homotyposis in Paramecium caudatum I 400–7
Honduras Piebald IX 330–31
Index Distributions, remarks on constants of VII 531–41
Inheritance of Intelligence XII 367–72
— Mathematical Theory of Determinantal VI 80–93
— of Meristic characters, note on III 363–65
— of psychical characters III 131–90, XII 367–72
Intelligence, relationship to head measurements and other physical and mental characters V 105–46
Limits to Area of Tails of Frequency Distributions, Generalised Tchebycheff Theorems XII 284–96
Mean square contingency, probable error of X 570–73
Mendelians views of law of ancestral inheritance III 109–12
Moriori Crania, question of artificial deformation of XIII 338–47
Multiple and Partial Contingency, general theory XI 145–58
— Correlation, errors with regard to X 181–87
Note on attack of Prof. A. von Török on use of arithmetic mean in craniometry II 339–45
— on the separate inheritance of quantity and quality in cow's milk VII 548–50
— on Dr Simpson's memoir on Paramecium caudatum I 404
Notice of paper on Past Experience and Future Expectation (reference) V 478
— — Tuberculosis (reference) V 478
Opsonic Index VII 533–41, VIII 203–24
Phthisis Death-rate XII 374–76
Probable error of Biserial η XI 292–301
— — in Multiple Normal Correlation. Determination of frequency constants of Normal distribution from moments of tail. Incomplete Normal moment functions VI 59–68
— — of coefficient of mean square contingency X 570–73
— — of tetrachoric r IX 22–27
Probability that two independent frequency distributions are samples from same population VIII 250–54, X 85–143
Ranks and Grades Correlation X 416–18
— — — application to examination results XIII 302–5

Pearson, Karl (*cont.*)
Reply to criticisms of G. U. Yule V 470–76
Scale order, certain points connected with, in the correlation of two characters which for some arrangement give a linear regression line V 176–78
Selection, influence of on variation and correlation VI 111–12, VIII 437–43
Sex-ratio, remarks on Cobb's note VI 109–11
Sexing of osteometric material X 479–87
Sickness table (reference) II 260–72
Significance of difference of sample and subsample V 181–83
Skew regression line, method of determining successive terms XIII 296–300
— variation, rejoinder to Ranke and Greiner IV 169–212
— — rejoinder to Kapteyn V 168–71
Skin-colour of crosses between Negro and White VI 348–53
Smallpox, distribution of severity of attack IV 505–10, V 362–64
Split-foot and lobster-claw, inheritance of VI 69–73, IX 381–90
Stability of offspring population, conditions for X 175–78
Tadpoles, biometric study of blood corpuscles in VI 402–19
Terns' eggs, co-operative study XII 308–54
Terns' eggs, co-operative study XV 294–345
Trypanosome strains X 85–143
Variation in sparrow's egg I 256, 257
White-lock family, Barton XIII 347–49
See also Editorial

Pearson, Karl and A. Barrington
Inheritance of coat-colour in cattle IV 427–64

Pearson, Karl, A. Barrington and A. Lee
Inheritance of coat-colour in greyhound III 245–98

Pearson, Karl and M. Beeton
Inheritance of duration of life I 50–89

Pearson, Karl and J. Blakeman
Probable error of mean square contingency V 191–97

Pearson, Karl and B. M. Cave
Variate Difference Method X 340–55

Pearson, Karl and A. G. Davin
Sesamoids of the knee-joint XIII 133–72
— — — evolution of XIII 350–400

Pearson, Karl and E. M. Elderton
Diphtheria attack and death-rate, influence of isolation on X 549–69
General Health, heredity character of IX 320–29
Natural Selection in Man X 488–506
Variate Difference method XIV 281–310; see also X 488–506

Pearson, Karl and D. Heron
On Theories of Association IX 159–316

Pearson, Karl, S. Jacob and A. Lee
Inter-racial correlation, cranial and head measurements II 347–56

Pearson, Karl and A. Lee
Inheritance in man, physical characters II 357–462

Pearson, Karl and M. A. Lewenz
Measurement of capacity from cranial circumferences III 366–97

Pearson, Karl and G. McMullan
Split-foot, inheritance of IX 381–90

Pearson, Karl and E. Pairman
Moment coefficients, corrections for XII 231–58

Pearson, Karl and E. S. Pearson
Polychoric coefficients of correlation XIV 127–56

Pearson, Karl and Marion Radford
Homotyposis in Beech leaves III 104–7

Pearson, Karl, H. E. Soper, A. W. Young, B. M. Cave and A. Lee
Small samples, distribution of correlation coefficient in XI 328–413

Pearson, Karl and P. Stocks
Unusual case of digital anomaly (with plates) XIV 410, 411

Pearson, Karl, E. Y. Thomson and J. Bell
Wasps VII 48–63, VIII 1–12

Pearson, Karl and J. F. Tocher
Differential death-rates, criteria for existence of XI 159–84

Pearson, Karl and W. F. R. Weldon
Inheritance in Phaseolus vulgaris II 499–503

Pearson, Karl, A. Wright and A. Lee
Wasps V 407–22

Pearson, Karl and A. W. Young
Probable error of Contingency coefficient without approximation XI 215–30, Corrigenda XII 259–60
Product moments, normal correlation surface XII 86–92

"Peccavimus," Editorial
Probable error of contingency coefficient without approximation. Corrigenda XII 259–60
On the conditions under which the probable errors of frequency distributions have a real significance, corrections and additions XII 260–66
Probable error and frequency distribution of errors in mean and S.D. of arrays XII 267–81
Corrigenda to 2nd Supplement to Memoir on Skew Variation XII 281

Pectinatella magnifica
statoblasts of I 128

Pedigrees
Albinism VII 243
Lobster-claw VI to face 71, IX to face 381
Piebald Family X to face 200
Pigmentation of human iris VIII 276
Split-foot VI to face 27
White-lock family, Barton XIII 347

Peile, W. H. (reference)
Whitechapel crania III 194, 196

Pelvis
Abac for determining obstetric conjugate IX 539–40
Correlation of measurements of IX 486–529
Variation in Female, American Indians (with plates) IX 34–57

Perrin, Emily
Contingency between occupation of father and sons III 467–69
Extrapolation III 99–103
Persians
Head measurements IV 295
Personal Equation V 472–76
Effect of in sexing crania and in taking measurements IX 393–98
Variations in and correlation of successive judgments XIV 23–102
Persons, W. M. (reference)
Variate difference method, Reply to criticism *re* correlations of X_p and X_{p+s}, X_p and Y_{p+s} etc. XIV 282–309
Petrie, W. M. Flinders (reference)
Naqada Crania I 411, 412, 422
Notice of lecture on Migrations V 480
Pfitzner, W. (references)
Hair- and eye-colour III 464, 465
Hand measurements I 345–60
Sesamoids
XIII 133, 134, 137, 158, 163–67, 170, 171, 351
Pflüger, E. (reference)
Frog's egg V 161, VII 201
Phagocytic Counts. Opsonic Index
Frequency distributions (Greenwood and White) VI 376–401
— — of means of samples (Greenwood and White) VII 505–30
— — Constants of (Pearson) VII 531–41
Reliability of for diagnosis, etc. (Harvey and McKendrick) VII 64–78
Comparison of indices calculated by different observers (Harvey and McKendrick) VII 79–95
Mathematical and Functional Error (Pearson) VIII 204–24
Phaseolus
Inheritance in II 290–92, 499–503
Weight of seed and characteristics of plant IX 11–21, X 72–84
Philippines
Cranial measurements, nasal characters IX 395, 399–432
Phillips, E. F. (reference)
Variability in bees V 419–21
Phillips, E. F. and G. M. Rommel
Inheritance of size of litter in sows V 203–5
Philosamia cynthia (silkworm)
Variation and elimination in III 113–30
Phthisis
Death-rate, check to fall of XII 374–76
See also Tuberculosis
Physical characters
in man, inheritance of II 357–462
— — compared with that of psychical characters III 131–90
Physical Defects
and pigmentation VI 197
Physical Development
Oxford Undergraduates VIII 44–47
Piebalds
Honduras, note on with plates IX 330–31
Negro (with plate) X 193
Piebalds (*cont.*)
Nyasaland Natives (with plates) IX 347–52
Cockayne's Family (with plates and pedigree) X 197–200
Pig
Fecundity in sows VI 433–36
Inheritance of size of litter in sows V 203–5
Sex-ratio and related phenomena in pig litters XV 373–81
Pigmentation
Anomalies among Nyasaland Natives IX 333–65
Colour of cuckoo's eggs ... I 164–76
Eye-colour, inheritance of (with plates and pedigree) VIII 267–79
Examples of anomalies IX 338–46
of Scottish Insane V 333–44
Piebalds and spotlings IX 347–52
Hair pigmentation anomalies IX 352–53
Leucoderma IX 353–58
Mesoblastic pigmentation in foetal eyes (with plates) XIII 51–56
Relation of Stature and Weight to VIII 340–53, 369–75
Skin-colour of crosses between Negro and White VI 348–53
Survey of Schoolchildren in Scotland, VI 129–235. Appendix of tables, etc. Supplement to VI 1–67
and susceptibility to disease VIII 13–39, 354–69
See also Albinism, Coat-, Hair- and Eye-colour, piebald
Pinnipedia
sesamoids in XIII 355
Pinus silvestris
Variation in length of needles I 21–23
Plots
Arrangement of, to secure most efficient results in testing cereals XV 271–93
Poisson
Exponential Binomial limit in connection with compound frequency distributions XI 139–44
Explanation of deviation from Poisson's Law in practice XII 211–15
Law of Small numbers. Table of exponential limit, X 25–35. Theory and application to cell frequencies, X 36–47. Criticisms of applications of, X 48–71
Normal Curve (reference) IV 189–93
Poles
head measurements IV 165, 295
Poll, H. (reference)
Moriori Crania XIII 339–45
Polychoric Functions
properties of, etc. XIII 409–17
Polychoric Table
Correlation coefficient, see under Correlation coefficient
Polydactyly
Nyasaland natives (with plates) X 18, 19

Polymorphism
Sources of apparent, in plants I 304–19, II 113–14, 114
Polynomial Function
Standard deviations of adjusted values of and guidance they give in choice of distribution of observations XII 1–85
Poppy
inheritance in II 56–100, (with plate) IV 394–426
Variability in VII 227–30
Popularity
inheritance of III 152, 155, 186
intelligence and V 126–31, 136, 146
Population
density of and pigmentation VI 185, 186
Powers and Sums of Powers
of Natural Numbers. Tables II 474–80
Powys, A. O.
Anthropometric data from Australia for the problem of evolution in Man I 30–49, IV 233–85
Fertility in Man (reference) XIV 337–45
Prediction Formulae
Brain-weight from head measurements IV 146–49
Capacity from cranial circumferences III 370–74
— — external measurements X 188–93
See also Reconstruction formulae
Primates
sesamoids in XIII 360–65
Prizes
most suitable proportion between the values of first and second I 385–99
Probabilities
problem of inverse IX 69–90, XIII 1–16
Probability
Integral, Sheppard's Tables of II 174–90
Limit to probability of deviations of given sizes without assumption as to distribution, Tchebycheff generalised limits XII 284–96
Probable Error
of Binomial Frequency constants X 37–38
of Class-Index Correction, Broad Categories XIV 261–75
— — — Quantitative Method XIV 275–80
of Contingency, coefficient of mean square V 191–97, X 570–73
— — — — without approximation XI 215–30
— — — — Corrigenda XII 259–60
of Correlation coefficient IX 4–6
— — Abac for VII 410, 411
— — Biserial X 384–90
— — Polychoric Table, Enneachoric Method XII 97–106
— — — — Polychoric Method XII 115–19
— — Ranks and Grades method, experimental determination XIII 264–82
— — Small samples, experimental work and empirical formula (Student) VI 302–10
— — — — P.E. to a 2nd approximation (Soper) IX 91–115
Probable Error (*cont.*)
of Correlation coefficient, Small samples (Fisher) X 507–21, XI 328–413
— — Tetrachoric IX 22–27
of Correlation Ratio, Biserial XI 292–301
— — — Table to assist computation of XI 302
of counting with haemacytometer V 351–60
of Criterion κ_2 for frequency types (Table) VII 396–97
of difference between Correlation coefficient and Correlation ratio IV 332–50
of difference of sample and sub-sample V 181–83, 315–33
of distance of mean of group from mean of whole sample in broad categories XIV 262–64
of Frequency constants, one variate. Moments, mean, correlation between errors in moments II 273–81
— — two variates, Correlation coefficient, product moments, means of arrays from regression line, coefficients of regression line IX 1–10
— — determined from grades, with special reference to medians, quartiles and percentiles XIII 113–32
— — tables to facilitate computation of, Probable error of β_1 and β_2, of skewness, of distance from mean to mode, Values of higher β's in terms of β_1 and β_2 VII 128–47
— distributions, significance of. Corrigenda for paper XII 261–66
of Mean II 274
— small samples from Normal population VI 1–25, X 522–29
— — — table for estimating probability that a mean lies between $-\infty$ and a given distance from mean of population VI 19, XI 414–17
— and Standard deviation of arrays XI 239–61, XII 267–81
of Mendelian Class Frequency, criticism of paper by Pearl XI 429–32
in Multiple Normal Correlation VI 59–68
Multiple Skew Regression XI 185–90
of Partial Correlation Coefficient VII 411–12
of Skewness and modal divergence, calculation of, for Normal Curve V 190
of small frequencies, application of law of small numbers to determine X 40–47
of standard deviation, points concerning, significance of assuming normality in practice when actual distribution is not normal, effect of Sheppard's correction VI 112–17
of Standard Deviations and fraternal and parental correlation coefficients XIV 1–22
Tables for facilitating computation of χ_1 and χ_2 IV 385–93
Variate Difference Method, mth difference correlation (Anderson) X 278
— — — note on (Ritchie-Scott), when population is kept constant by taking in an additional observation at each successive difference XI 136–38
Table of $2\phi(m) - 1$ for $m = 1$ to 20 XI 138

Product moments
Correlation between product moments of different orders IX 1–3
— — product moment about a fixed axis and one about an axis through the centroid IX 6–9
Formulae requisite for the determination of probable errors in the case of two variables IX 3
of Normal correlation surface of two variates, Tables of in terms of correlation coefficient XII 86–92
of Normal distribution in n variables XII 177–83
of any order Normal distribution in any number of variables, formulae for XII 134–39. Corrigenda, XII 266
Standard deviations of product moments of any order IX 1–3
See also Frequency constants
Protective Colouring, see Colour
Protozoa I 400–7, II 241–54, 321–37, V 53–72, 213–97
Prussians
Age and hair- and eye-colour III 462–65
Hair- and eye-colour correlations III 459, 461
Pseudoclytia pentata (jellyfish) I 255
Psychical Characters
Inheritance of, Heymans and Wiersma data V 460–69
— Pearson's data for schoolchildren III 131–90
— Comparison of method of Broad categories and Binet-Simon tests for Intelligence XII 367–72
Correlation with Intelligence in Schoolchildren, Pearson's data V 126–36
See also Intelligence
Psychophysical Curves
Criterion of Goodness of fit of XII 216–30
Pulmonaria officinalis (lungwort)
Variation in flowers and heterostylism III 398–458
Pulse
Association with physique and intelligence in criminals XI 1–19
and Breathing rates and Stature VIII 232–36
Punnett, R. C.
Brachydactyly (reference) VI 118, 327
Merism and sex in Spinax niger III 313–62
Pure Line
Johannsen II 499–503
Hanel, Jennings, Johannsen, Warren VII 370–81
Purity of Gametes I 228–54, II 44–55, 286–98, III 1–51
See also Mendelism

Quadrature Coefficients
Table of, for Sheppard's (c) formula XII 282, 283
Quadrature Formulae I 272–282
Application of IV 378, 379
Quakers
Assortative-mating among II 481–98
Inheritance of duration of life I 50–89
Qualitative Variates, see Broad categories
Quatrefages, A. de (reference)
Crania XI 100, XIII 338, 340, XIV 220

Rabbit-breeding II 299–306
Race
Inter-racial Variation and Correlation between Man and Woman IV 162, 286–312
Racehorses
Inheritance of sex-ratio in VI 120–22
Races of Man
Comparison of Cranial measurements, Coefficient of racial likeness XIII 247–51, XIV 205–27, 248, 249
— — — Burman and adjacent races XIII 235–51
— — — English and other races III 208–13, 221–26, 231–39
— — — Moriori and other races XI 94–133
— — — Naqada and other races I 424–40
— — — Negro and other races VIII 298–324, 333–36
— — — Tibetan and other races XIV 203–49
— — type contours VIII 131–38, XI 103–12, XIV 227–48
— races for nasal characters IX 409–17
Inter-racial prediction formulae for capacity from cranial circumferences III 371, 380–90
See also Inter-racial correlation
Racial Heredity
Jews and Gentiles VIII 424–30
Radford, M. and K. Pearson
Homotyposis in beech leaves III 104–7
Rana (frog)
Symmetry of egg and symmetry of embryo V 147–67, VII 148–209
Rana temporaria (tadpole)
Study of red-blood corpuscles of VI 402–19
Random Occurrences
in space and time XIII 309–37
Random Sampling
On the test for I 155
Ranke, J. (reference)
Cranial measurements I 413–15, III 206, 226, IV 169, 170, 172
Ranke, K. E. (reference)
Skew Variation IV 169–212
Ranks and Grades (Method of Correlation)
Application to Examination results XIII 302–5
Correction for ties, Experimental determination of Probable Error XIII 264–82
Relation between actual variate correlation and correlation as determined from Ranks or Grades X 416–18
Rats
Fertility and weight in VII 381–85
Reaction Times
Degree of association between age, acuity and reaction times for different senses XV 346–72

Reconstruction Formulae
Capacity from cranial circumferences III 370–74
Stature from other characters I 209–13
See also Prediction formulae
Regeneration
of feet in Blattidae VI 311–26
Regression
I 322, 323, 365–74, II 217–28
Curves, goodness of fit XI 239–61
General forms of bivariate frequency distributions, when regression and variation are subjected to limiting conditions XV 77–88, 209–21
Line, probable error of coefficients of IX 9
— skew, method of determining successive terms XIII 296–300
Linearity of, tests for IV 332–50
Multiple VIII 239
— skew, probable errors in XI 185–90
Non-linear, three or more variables X 391–411
Reid (reference)
Brain-weights III 63, 68, IV 31, 129, 133
Reproductive selection, see Selection
Reptiles
sesamoids in XIII 389–90
Resemblance
between Man and Woman IV 161–8
Retzius, A. (reference)
Brain-weight IV 17
Reversion II 172, 289
See also Mendelism
Reviews
Frequency Arrays (H. E. Soper) XIV 418, 419
Some Formulas in the Theory of Interpolation of Many Independent Variables (S. Narumi) XIV 420
Study of Women Delinquents in New York XIII 305–8
Tuberculosis as an acute disease of Childhood (K. F. Andvord) XIV 191, 192
Rhind, A.
Frequency distributions, errors of type VII 386–97 (tables)
Probable errors, Tables for computation of, Probable errors of β_1 and β_2 etc. VII 128–47
Rhodes, E. C.
Skew Correlation Surface XIV 355–77
Time intervals between births XV 412–21
Ridgeway (reference)
Anthropometry of inhabitants of Eastern Mediterranean XIII 93
Rietz, H. L.
Inheritance in Cows VII 106–26
Ripley, W. Z. (reference)
Anthropometry of inhabitants of Eastern Mediterranean XIII 93, 109–11
Ritchie-Scott, A.
Correlation Coefficient, Polychoric Table XII 93–133
— — — (reference) XIV 127–56
Incomplete moments of Normal Solid XIII 401–25
Ritchie-Scott, A. (*cont.*)
Probable error of Correlation Coefficient in Variate Difference Method, note on XI 136–38
Robbins, W. W.
Gaillardia aristata, variation in VI 106–8
Rock, F.
Teeth and use of Tooth brush VIII 237, 238
Rodents
sesamoids in XIII 357–59
Romanovsky, V.
Moments of binomial about its mean XV 410–12
Rommel, G. M. and E. F. Phillips
Inheritance of size of litter in sows V 203–5
Ross, R. (reference)
Malaria and Mosquitoes VII 421–36
Round Barrow Crania
Measurements and descriptive catalogue IV 351–62
Roux, W. (reference)
Egg and embryo of frog V 147, 148, 157, 161–67, VII 201, 204, 206
Rowan, W.
Co-operative Study on Terns' Eggs X 144–68, XII 308–54
Royal Engineers
Head measurements and cranial type contours VIII 131–33, 197–201
Russians
Head measurements IV 161–68, 286–312
— — Pigmentation and stature XV 254–70
Ryley, K. V. and J. Bell
Study of Nasal Bridge in Anthropoid Apes IX 391–437

Salaman, R. (reference)
Mendelism and racial heredity VIII 424–30
Saltzmann (reference)
sesamoids XIII 150
Samples
Curves for describing frequency of random samples V 172–75
Probability that two independent frequency distributions are samples from the same population VIII 250–54, X 85–143
Test for significance of difference of sample and sub-sample V 181–83
— — Application to Scottish Insane data V 315–33
Small, see Small samples
Sampling
Probable occurrences in second small samples (with tables) IX 69–90
Sanguinaria (bloodroot)
Morphology of fruit of VII 305–51
Sassafras
Variation and Distribution of leaves XII 372, 373
Saunders, A. M. Carr
Pigmentation, selection and anthropometric characters VIII 354–84
Saunders, E. R.
Mendelism II 44–55
Scale order
effect of changing, on correlation V 176–78

Scarlet Fever
and Pigmentation (Macdonald) VIII 22–39
— — (Saunders) VIII 360–65
Schoolchildren
Association of Drawing with other capacities VII 222–26
Inheritance of Intelligence. Gordon and Drinkwater data, etc. XIV 378–408
— Psychical characters in. Pearson's data III 131–90
Intelligence and head measurements and other physical and mental characters. Pearson's data V 105–46
— and home conditions XV 161–72
— Teachers' estimate VII 542–48, VIII 79–93, 94–108
Pigmentation of, in Scotland VI 129–235
Supplement to VI 1–67
— Stature and Weight (Elderton) VIII 340–53
— — — (Saunders) VIII 369–77
— and susceptibility to disease VIII 13–39
Stature and Weight. Glasgow data X 288–339
Schtscherbakow, Th. S.
Blattidae, four-jointed tarsus and regeneration of feet in VI 311–26
Schulze, O. (reference)
Egg and embryo of Frog V 148, 163, 167
Schuster, E. H. J.
Ancestral Inheritance, law of and Prof. Ziegler V 184
Anthropometric Laboratory, Oxford VIII 40–51
Crabs, variation in Eupagurus prideauxi II 191–210
Hereditary Deafness IV 465–82
Long and Round Barrow Crania IV 351–62
Mice, crossing of grey and albinos IV 1–12
Vital capacity, height, weight, etc. (reference) XIV 323–26
Schuster, E. H. J. and E. M. Elderton
Inheritance of Psychical characters V 460–69
Scotland
Pigmentation Survey VI 129–235. Supplement 1–67
Scott (reference)
Crania XI 82, 92–97, 104, XIII 340, 341, 344, 345
Scottish
Cranial measurements, comparison with English, Whitechapel III 241, 242
— — — Negro VIII 321–24
Insane, anthropometry of V 298–350 and supplement
Schoolchildren, Pigmentation and stature and weight VIII 340–53
— — and susceptibility to disease VIII 13–39
— stature and weight of, in Glasgow X 288–339
Sea-cucumber (Holothuria)
Variation, development and growth VI 236–301
Seasonal polymorphism
in plants I 304–19, II 113, 114
Seasonal variation
Termites VI 334, 335
Terns' eggs XII 310, 311
Wasps VIII 6–8
Seed
influence of weight of on plant IX 11–21, X 72–84
Segregation
influence of on Tuberculosis X 530–48
Law of I 228–54
Selection
Ancestral Heredity, law of II 224–28
Influence of in modifying intensity of inheritance in poppies IV 403–8
— on correlation and variation VI 111, 112, VIII 437–43
— on heredity in cows VII 113–17
— on inheritance in foxgloves XIV 103–26
Intermediates in Bacillus coli IX 331, 332
Pigmentation and VIII 13–39, 354–69
Selective elimination during development of fruits of Staphylea VII 452–504
Sexual selection in man IV 161–68, 287, 292
Selection, natural
in Clausilia laminata I 109–24
in Lepidoptera III 113–30
in Man, correlation of death-rates in infancy and childhood X 488–506
— intensity of IX 58–68
— — among the English I 50–89
— Snow's reply to critics VIII 456–60
in snails V 387–99
in wasps VIII 6–12
Selection, periodic
in Clausilia laminata I 119–24
in Man II 371, 395
in snails III 299–306
Selection, reproductive X 419, 420
in Man II 371–73, 395, 396, IV 280–85
in mice V 441–42
Self-consciousness (introspection) III 152, 155, 185, V 126–31, 134, 135, 146
Sergi, S. (reference)
Inhabitants of Eastern Mediterranean XIII 93
Sesamoids (of the knee-joint)
Part I. Man (with plates) XIII 133–72
Part II. Evolution of sesamoids (with plates) XIII 350–400
Addendum XIV 409
Severity of cases of disease
means of estimating V 423–34
distribution of severity of attack in smallpox IV 505–10, V 361, 362, 362–64
Sex
Brain-weight and IV 45–51
Dimorphism in Spinax niger III 318–21
Heredity in VI 109–11
Influence of number of changes of on inheritance II 237–40, III 257
Inter-racial correlation and variation between the sexes IV 161–68
and merism in Spinax niger III 313–65
Sex-ratio
and related phenomena in pig-litters XV 373–81

Sex-ratio (*cont.*)
Note on by Cobb VI 109
— — Pearson's remarks on VI 109–11
Inheritance of, in man V 73–78
— — — and horses V 79–85
— in mice V 436–49
— in racehorses VI 120–22
in Spinax niger III 321
Sexing
Mathematical sexing of osteometric material X 479–87
— — Dwight's method X 209–12
Personal Equation, effect of IX 393–98
Sexual selection, see Selection
Sheppard, W. F.
Corrections for moments I 287, V 450–59 (reference) III 308–12, VIII 443
Moments, calculation of V 450–59
Quadrature Formulae and curve fitting (reference) I 273–279
Table of Deviates of Normal Curve V 405
— of Probability Integral II 174–90
Sheppard's Corrections
for Moments V 450–59
— Elementary Proof of III 308–12
— Fallacious Proof of VIII 443
Sheppard's Tables
of Probability Integral II 174–90
Shorthorns
Inheritance of coat-colour in IV 427–64
Shrimps V 14, 16, 17
Shrubsall, F. C. (reference)
Crania VIII 292–96, 302
Shufeldt, R. W. (reference)
Sesamoids XIII 386, 387
Shull, G. H.
Asters I 311, II 113
Shyness (assertiveness)
Inheritance of III 152, 155, 184
Intelligence and V 126–31, 134, 135, 146
Sickness-table
Graduation and Analysis of II 260–72, III 52–57
Addendum II 503, 504
Significance of difference
of death-rates, criteria for XI 159–84
of sample and sub-sample V 181–83, 315–33
See also VIII 250–54, X 85–143, on the probability that two independent frequency distributions are samples from same population
Silkworms (Philosamia cynthia)
Variation and elimination in III 113–30
Simpson (reference)
Quadrature Formula I 274
Simpson, J. J.
Statistical Study of Cruciferae X 215–68
Simpson, J. Y.
Paramecium caudatum I 400–7
Sims (reference)
Brain-weights IV 31, 129–33
Sium latifolium XI 47–49
Skeletal Measurements, see Anthropometry
Skew Correlation Surface (with plate) XIV 355–77
Skew Regression, see Regression
Skew Variation
Pearson's reply to Prof. Kapteyn V 168–71
Reply to criticisms of Ranke and Greiner IV 169–212
Second supplement to memoir on, Corrigenda XII 281
Skewness
Probable Error of VII 129, 144, 145
Skin-colour, see Albinism, Piebalds, Pigmentation
Skull
Browne, Sir Thomas, skull, portraits and ancestry (with plates) XV 1–76
Skulls, see Crania
Small Numbers, Poisson's Law
Table of Exponential Binomial limit X 25–35
Application to Cell Frequencies X 36–47
Criticisms of Applications of X 48–71
Explanations of Deviations from in practice XII 211–15
Small Samples
Frequency distribution of Correlation Coefficient in, Experimental work, etc. (Student) VI 302–10
— — — — Distribution to a second approximation (Soper) IX 91–115
— — — — Actual Distribution (Fisher) X 507–21
— — — — Determination of constants and ordinates and of most probable value of correlation in sampled population XI 328–413
— — of Standard Deviations of VI 3–6, X 522–29
Probable Error of Correlation Coefficient, Experimental work (Student) VI 302–10
— — — — To a second approximation (Soper) IX 91–115
— — — — (Fisher) X 507–21
— — — — (Co-operative Paper) XI 328–413
— — of a mean VI 1–25, X 522–29
— — — Table for estimating probability of deviations from mean of population VI 19, XI 414–17
— occurrences in second small samples (with tables) IX 69–90
Standard Deviations of samples of two or three, note on XI 277–80
Tables, see under Tables, Small samples
Smallpox V 431–35
Distribution of severity of attack IV 505–10, V 361–64
Influence of previous vaccination I 375–83, II 135–44
Vaccination IV 313–31, 483–504
Smith, G.
Relation of nucleus and cytoplasm II 241–54
Smith, H. D.
Crania, Egyptian, observation on occipital bone (with plates) VIII 257–61
— — Pigmy (with plates) VIII 262–66
Smith, Kirstine
"Best" values of frequency constants XI 262–76

Smith, Kirstine (*cont.*)
Choice of the Distribution of Observations XII 1–85
Fraternal and Parental Correlation Coefficients XIV 1–22
Snails V 32–34, 37, 38
Natural selection in I 109–24, V 387–99
Periodic selection in I 109–24, III 299–306
Variation in I 109–24, 468–92, II 24–43
Snow, E. C.
Application of Correlation Coefficient to Mendelian Distributions VIII 420–24
Natural Selection in Man, reply to critics VIII 456–60
— — — Second Paper IX 58–68
Somatic Characters
Correlation with fertility in "hibiscus" VIII 52–65
See also Ancestral heredity and Mendelism
Sömmering (reference)
Sesamoids XIII 151
Sommerville, D. M. Y.
Classification of Frequency ratios V 179–81
Soper, H. E.
Exponential Binomial Limit, Tables X 25–35
Frequency Arrays, Review XIV 418, 419
Probable Error of Biserial correlation coefficient X 384–90
— — of Correlation Coefficient, small samples IX 91–115
Small samples, Distribution of Correlation coefficients in (co-operative paper) XI 328–413
— — — — — (reference) X 507–21
de Souza, D. H.
Measurements of Pelvis with special reference to obstetric prediction IX 486–529
Sow, see Pig 97
Sparrow
variation in egg I 256, 257
Spasm, Facial
Inheritance of, with pedigree XIV 311–15
Spearman, C. (references)
Curve-fitting, psychophysical data XII 219
Hierarchical order among correlation coefficients XII 356–66
Ranks and Grades Method of Correlation XIII 264–82
Species
Investigation as to existence of sub-species in cuckoo IV 363–73
Origin of I 365–74
Spencer, J.
Graduation of sickness table III 52–57
Spherical Trigonometry
and Multiple Correlation XV 173–208
Spiegel, A. (reference)
Sesamoids XIII 147
Spinax niger
Sex and merism in (with plates) III 313–65
Spleen
weight of human III 63–83
Split-foot
and Lobster Claw (Pedigree and plates) VI 69–79, IX 381–90
Addendum IX 540
Split-foot (*cont.*)
and split-hand deformities (Pedigree and plates) VI 26–58, 117–18
— — Nyasaland Natives X 17
See also note on Brachydactyly VI 327
Spurious Correlation
Elimination of X 179–80
See also Variate Difference Method
Fictitious Association from Mixing of Records II 132–34
Stability
of offspring population, conditions for X 175–78
Standard Deviation and Standard Deviation of Arrays
See Frequency Constants, Frequency Distributions, Probable Error
Stannus, H. S.
Congenital Anomalies in Native African Race X 1–24
Pigmentation Anomalies among Nyasaland Natives IX 333–65
Staphylea (American Bladder-Nut)
Selective elimination during development of fruits VII 452–504
Statoblasts
of Pectinatella magnifica I 128
Stature
and Age, Australian Criminals I 38–49
— Glasgow schoolchildren X 288–339
and Brain-weight IV 58–64, 115–23
of Criminals I 38–49, 192–202, 209–13
of Egyptians VIII 66–78, XI 67–81
Family Measurements, relation to span and forearm, inheritance II 357–462
of Inhabitants of Eastern Mediterranean XIII 104, 105
Inter-relations of Vital capacity, stature, stem length, weight, etc. XIV 316–36, XV 109–33
of Oxford Undergraduates VIII 45, 46
and pigmentation VIII 340–53, 369–75
reconstruction of from other characters I 209–13
relation to pulse and breathing rates VIII 232–36
of Scottish insane V 298–350 and Supplement
See also Anthropometry
Steadiness and rapidity of hand
Association with Artistic Capacity XII 170–77
Stephens, J. W. W. (reference)
Trypanosome Strains X 89–97
Stevenson, T. H. C.
Social distribution of mortality from different causes in England and Wales, 1910–12 XV 382–400
Stocks, P.
Facial spasm inherited through four generations XIV 311–15
Inheritance of cranial anomalies XV 406–10
Stocks, P. and K. Pearson
Unusual case of Digital Anomaly (with plates) XIV 410–11

Strangeways, T. S. P. (reference)
Opsonic Index VI 380–401, VIII 207–15
Student
Correlation Ratio, correction for grouping IX 316–20
Error of counting with haemacytometer V 351–60
Explanation of deviations from Poisson's Law in practice XII 211–15
Frequency Distributions, Means of samples not drawn at random VII 210–14
Ranks and Grades Correlation Method, Correction for Ties, Experimental Determination of Probable Error XIII 264–82
Small samples, distribution of correlation coefficient (reference) X 507, 508, 512
— — probable error of correlation coefficient VI 302–10
— — — — of mean VI 1–25
— — — — — Table for estimating probability of deviations from Mean of population XI 414–17
Spurious Correlation, elimination of, Variate Difference Method X 179, 180
Testing varieties of cereals XV 271–93
Variate Difference Method (reference) X 340, 489, XIV 285, 289, 300, 306, 309
Sub-sample
test for significance of difference of, from sample from which it is drawn V 181–83, 315–33
Sulman, P. L.
Terns' eggs, co-operative study XII 308–54
Summation Method
of calculating moments IV 374–78
Surface, F. M.
Fecundity in Sows VI 433–36
Susceptibility
to disease and pigmentation VIII 13–39, 354–69
Sutherland, G. F. (reference)
Criminals VII 232, 233
Swedes
Brain-weights and other anthropometric characters IV 13–104, 133, 134, 135
Cranial Measurements, compared with English, Whitechapel III 243
Hair- and eye-colour correlations III 459, 461
Swine, see Pig
Sylvius, Jacob (reference)
Sesamoids XIII 145
Symmetry
in crabs II 307–20
in John Dory II 115–20
in jellyfish I 255, 256
of egg and embryo in frogs V 147–67, VII 148–209
Principle of, Bateson I 324
Syphilis
Attempt to ascertain the prevalence of, in a large urban population (reference) VIII 437

Tables
Corrective terms for moments of trapezia II 9
Errors of Type of Frequency Distributions, determination of VII 390–97
Gamma Function, Log $\Gamma(p)$ from $p=1$ to $p=2$ VII 43–47
Goodness of Fit, tables for testing I 155–63
Integral

$$q_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\phi\, d\phi = \int_0^{\frac{\pi}{2}} \cos^{n-1}\phi\, d\phi \quad \text{XI 377}$$

Mesodacryal Indices VIII 338, 339
Supplementary Tables IX 401–2
Moment Functions, Incomplete Normal VI 66, 67
Normal Curve, Areas and Ordinates, Sheppard's Tables II 174–90
— — Deviates of, for each permille of frequency V 405
— — Ordinates of, for each permille of frequency XIII 428
— — values of ψ_1 and ψ_2 to determine constants from moments of Tail, VI 68; when Tail is larger than body, X 214
Occipital Index, table for calculating XIII 261, 262
Poisson-Exponential for cell frequencies 1 to 30 X 45–47
Poisson's Exponential Limit Tables of $e^{-m}m^x/\underline{x}$ X 27–35
Powers and Sums of Powers of Natural Numbers 1 to 100 II 474–80
— of Natural Numbers 1 to 20, sums of II 10
Probable Error, Tables of χ_1 and χ_2 IV 385–93
— — of Biserial r X 389
— — of Biserial η. Values of ψ' and ψ'' XI 302
— — of frequency constants, β_1 and β_2; Probable error of skewness and of distance from mean to mode, Values of higher β's in terms of β_1 and β_2 VII 128–47
— — of Tetrachoric r. Tables of χ_r and χ_a IX 26, 27
Probability Integral, Sheppard II 174–90
Product Moments, high order normal XII 90–91
Quadrature Coefficients for Sheppard's Formula XII 283
Small Samples, Correlation in, Ordinates and Constants of Frequency Curves XI 379–404
Small Samples, Correlation in, To assist calculation of Ordinates of Correlation Frequency Curves from expansion formulae XI 405–11
— — — Position of Origin and Abscissal Unit in terms of standard deviation XI 412–13
— — Constants of Frequency distributions of Standard Deviation X 529, XI 279

Tables (*cont.*)
Small Samples, Probable error of Mean, Tables of

$$\frac{n-2}{n-3}\cdot\frac{n-4}{n-5}\cdots\begin{pmatrix}\frac{3}{2}\cdot\frac{1}{2}\,n\text{ odd}\\ \frac{2}{1}\cdot\frac{1}{\pi}\,n\text{ even}\end{pmatrix}\int_{-\frac{\pi}{2}}^{\tan^{-1}z}\cos^{n-2}\theta\,d\theta,$$

for $n=4$ to 10 VI 19
— — Probable occurrence in 2nd small samples IX 82–90
— — Tables for estimating probability that the mean of a sample lies between $-\infty$ and any given distance from the Mean of whole population XI 416–17
Tetrachoric Functions VII 442–51
— — Supplementary tables for high correlations VIII 390–95
— — — — — — — r negative XI 288–91
Variate Difference Method, Functions for Geometrical Decadence XIV 310
Tadpoles
Study of blood corpuscles in VI 402–19
Tail
Constants of Normal Frequency Distril tion determined from moments of tail VI 63–68
Functions, Normal, Table of VI 68
— — — when tail is more than half curve X 208–14
Upper and lower limits to the tails of frequency distributions XV 245–53, 421–23
See also Tchebycheff Generalised Limits
Tarnowsky, Pauline (reference)
Criminals VII 231, 232
Tatars
Head measurements IV 165, 295
Taylor, G. (reference)
Sesan oids XIII 378
Tchebycheff Generalised Limits
Limit to probability of deviations of given sizes without assumption as to distribution XII 284–96
Tchouproff, A. A.
Mathematical Expectation of Moments XII 140–69, 185–210, XIII 283–95
Teeth
and Use of Tooth Brush VIII 237, 238
Temper
Inheritance of III 152, 155, 188
Intelligence and V 126–31, 134, 135, 145
Temperature
Association with physique and intelligence, in criminals XI 1–19
Termites
Statistical observations on VI 329–47
Variation in, compared with wasps VII 53, 59
Terns
Homotyposis and allied characters in eggs of X 144–68 (with plates)
Nest and Eggs of (with plates) XII 308–54, XV 294–345
Association of Nest and Egg Pattern XII 311–18, XV 308–17
Terns (*cont.*)
Change of Type of Egg with Season XII 310, 311, XV 306–8
Homotyposis XII 331–35, XV 324–29
Organic Correlation XII 325–31, XV 317–24
Problem of Mixed Colour clutches XII 318–25
Tetrachoric Functions
Expansions in XIV 157–85
Tables VII 437–51
— Supplementary Tables for high correlations VIII 385–95
— — — — — — r negative XI 284–91
Tetrachoric Tables
Goodness of fit XIV 189–91
Thane, G. D. (references)
Determination of Inion III 376, 393
Sexing of Crania I 411, IV 351, IX 394, 396
Thebans
Cranial measurements, compared with Naqada I 426–65
— — — Negro VIII 306–20
Thiele, T. N. (reference)
Notice of Book on Interpolation VII 413
Thompson, H. (reference) I 419–21
Thompson, W. D'Arcy (reference)
Sesamoids XIII 388
Thomson, E. Y.
Crania, Moriori XI 82–135
(references) XIII 338–47, XIV 228, 232, 233, 240, 241, 246, 247
Cranial measurements XIII 179, 180, 216
Thomson, E. Y., J. Bell and K. Pearson
Wasps VII 48–63, VIII 1–12
Thomson, G. H.
Goodness of Fit of psychophysical curves XII 216–30
Hierarchical order among correlation coefficients XII 355–66, XV 150–60
Tildesley, M. L.
Association of Artistic Capacity and steadiness and rapidity of hand XII 170–77
Browne, Sir Thomas, his skull, portraits and ancestry (with plates) XV 1–76
Burmese Crania (with plates) XIII 176–262
Crania (reference) XIV 200, 228, 232, 250
Terns' eggs, co-operative study XII 308–54
Tocher, J. F.
Anthropometry of Scottish Insane V 298–350 and Supplement
Pigmentation Survey of Schoolchildren in Scotland VI 129–235 and Supplement
— comparison with children suffering from scarlet fever (reference) VIII 18–39
Tocher, J. F. and K. Pearson
Differential Death-rates, criteria for existence of XI 159–84
Török, A. von
Note on Cranial types II 508
(reference) Remarks on Note II 509–12
(reference) Note on his attack on the use of the arithmetic mean in craniometry II 339–45
Tower, W. L.
Chrysanthemum leucanthemum I 309–15
Trew, C. J. (reference)
Sesamoids XIII 148, 149

Trifolium pratense quinquefolium (clover) I 371–73
Troup, J. McD. and G. D. Maynard
Distribution of cases of disease in houses VIII 396–403
Trout
growth, variability and correlation in VIII 444–55
Trypanosome Strains X 85–143
Probability of identity of X 92–101
Influence of host in which trypanosome is cultivated X 101–19
Heterogeneity in measurements X 119–42
Tschepourkowsky, E.
Inter-racial Correlation IV 286–312
Resemblance between Man and Woman IV 161–68
Russian Anthropometry (with plate) XV 254–70
Tuberculosis
as an acute disease of childhood. Review XIV 191, 192
and Segregation X 530–48
See also Phthisis
Turks
head measurements IV 295
Turnbull, H. M.
Fowl, malformation in head of IX 538, 539
Hospital data for study of variability and weight of human viscera (reference) IX 475, 476
Turner, F. M.
Distribution of severity of attack in smallpox V 361, 362
Vaccination and smallpox IV 483–504
Turner, William (reference)
Crania III 192, 200, 241, 242
Typhoid
Anti-typhoid inoculation VI 366–75
Typhus V 424–30

Umbelliferae
Variation and correlation of number of umbel rays XI 38–49
Ungulates
sesamoids in XIII 353, 354
Urban, F. M. (reference)
Goodness of fit, psychophysical data XII 216–30
Usher, C. H.
Examination of Albinotic Eye XIII 46–56

Vaccination
and Smallpox I 375–83, II 135–44, IV 313–31, 483–504
Variate Difference Method
Fundamental Formulae (Student) X 179, 180
Standard Deviation of mth difference and Probable Error of mth difference correlation, when correlation has become steady (Anderson) X 269–79
Illustrations of Method, Application to Mortara's table of Index Values for Italy X 340–55
Variate Difference Method (*cont.*)
Application to correlation of death-rates in infancy and childhood X 488–509
Tests for "Steadiness" of difference correlations X 494–500
Note on Probable Error of Correlation Coefficient when population is kept constant by taking in additional observation at each difference XI 136–38
Table of $2\phi(m) - 1$ for $m = 1$ to 20 XI 138
Reply to criticism of Yule *re* swamping of random fluctuations by periodic term, Reply to criticism of Persons *re* neglect of correlation between X_p and Y_{p+s}, X_p and X_{p+s} etc. and modification of theory to cover this XIV 282–309
Table of Functions of Geometrical Decadence XIV 310
Application to correlation of successive judgments XIV 37–45
Criteria for distinguishing various types of residuals XV 134–49
Variation I 321–44
Continuous and discontinuous and Mendelism VIII 423
De Vries' Mutation Theory and I 365–74
Hanel on VII 373–81
influence of selection on VI 111, 112, VIII 437–43
in Adoxa VII 218–22
in Anemone, number of sepals I 307–9
in Aphis I 129–54
in Arcella II 321–37
in Aster II 114
in Aurelia aurita I 90–108
in Brain-weight IV 13–104, 124–60
in Carabus auratus, variation of sexes VI 103–5
in Celandine II 145
in Chilomonas in different environment V 53–72
in Cichorum intybus, L. V 184–87
in Earthworm IV 213–29
in Eupagurus prideauxi II 191–210
in Fowl, comb of domestic VI 420–32
in Foxgloves, of race and families XI 319–20
in Gaillardia aristata, flower heads of VI 106–8
in Helix nemoralis I 468–92
in Holothuria, variation, development and growth VI 236–301
in Human Hand I 345–60
in Human Skull I 408–66, III 191–245
in Jellyfish I 255, 256
in Length of needles in Conifers VI 354–65
in Lepidium draba, flowers of X 215–68
in Lepidoptera, and elimination III 113–30
in Man, Table of coefficients of variation for various head and body measurements IV 32
in Meristic series, Spinax niger III 321–38
in Moschatel II 108–13
in Mulberry tree, leaves of I 258, 259
in Needle-Length of Conifers VI 354–65

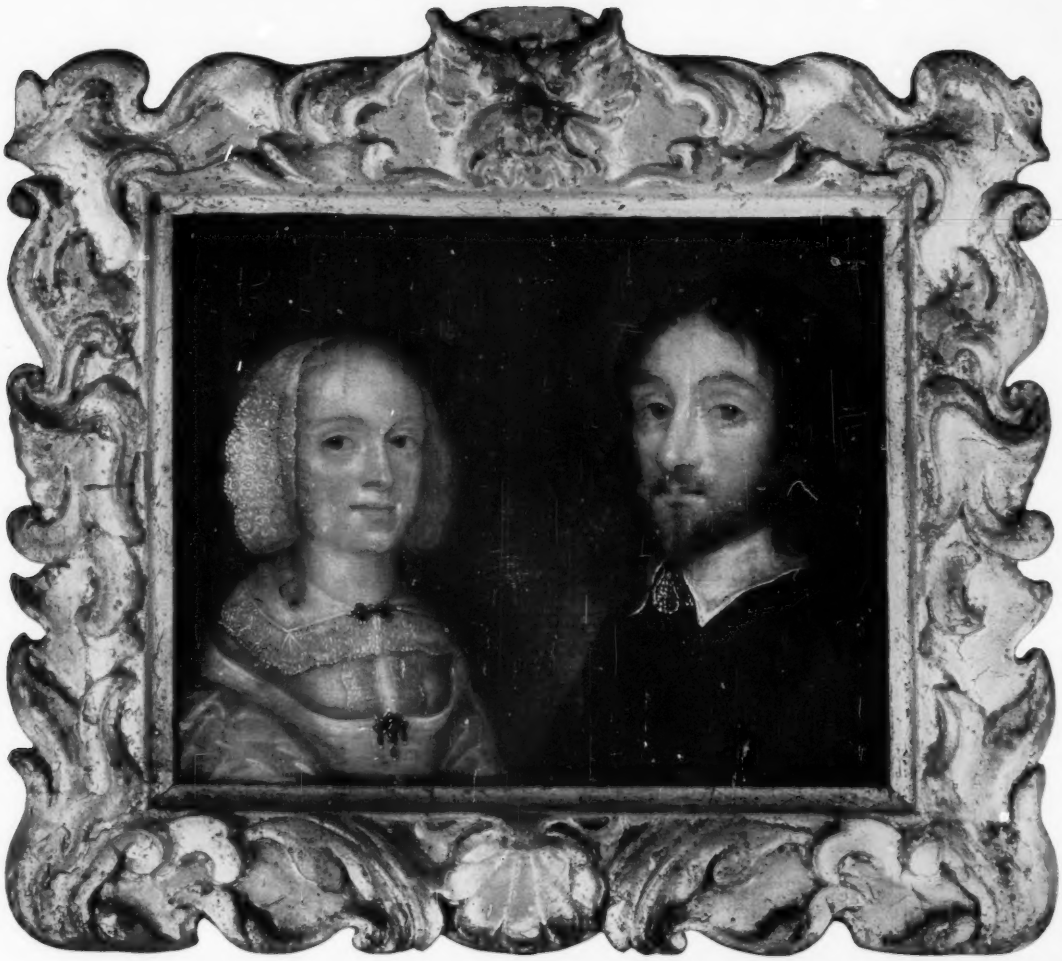
Variation (*cont.*)
in Ophiocoma nigra II 463–73
in Paramecium V 224–39
in plants I 11–29
in Poppies, Shirley VII 227–30
in Pulmonaria officinalis III 398–458
in Sassafras, leaves of XII 372–73
in Scottish Insane V 298–350
in Snails V 387–99
in Sparrow's egg I 256, 257
in Termites, VI 329–47; compared with that in Wasps, VII 53, 59
in Trout VIII 449–55
in Umbelliferae, number of umbel rays XI 38–49
in Wasps, clypeal markings X 201–7
— wings of V 407–22, VII 48–63
in Weight of human Viscera III 63–83, IX 473–85
Veddahs
Cranial measurements, nasal characters IX 395, 399–407, 410–21, 425, 427, 432
Velde, G. (reference)
Anthropometry, Inhabitants of Eastern Mediterranean XIII 94, 95
Venn, John (reference)
Problem of inverse Probabilities XIII 2, 3
Verschaffelt (reference)
Correlation I 125–28
Vertebrates, see Cattle, Dogs, Frogs, Horses, Mice, etc.
Vesalius (reference)
Sesamoids XIII 145–47
Vespa vulgaris, see Wasps
Viscera
Weight, variability and correlation of human III 63–83, IX 473–85
Vision
Acuity of in Oxford Undergraduates VIII 43
— of, and Reaction Times XV 346–72
Vital Capacity
Inter-relations of vital capacity, stature, stem length, weight, etc. XIV 316–36, XV 109–33
See also Lung capacity
Vivacity (noise)
Inheritance of III 152, 155, 183
Intelligence and V 126–31, 134, 135, 146
De Vries (reference)
Hairiness of Lychnis II 47–49
on the origin of species I 365–74

Waite, H.
Finger-prints, Association of X 421–78, (reference) XIII 61–63
Intelligence, Teachers' Estimate VIII 79–93, (reference) XIV 383, 384
Malaria and Mosquitoes VII 421–36
Warren, E.
Aphis, variation and inheritance in I 129–54
Cranial Measurements (reference) I 411, 435, 463
Daphnia magna II 255–59
Warren, E. (*cont.*)
Foxgloves, Breeding experiments with XI 303–27
— Inheritance in XIV 103–26
Parthenogenesis (reference) I 400
Tadpoles (reference) VI 402–19
Termites, Statistical observations on VI 329–47
— comparison of observations with those on wasps (reference) VII 53–59
Wasps
Average length of absence from nest V 365–86
Clypeal markings X 201–7
Comparison of queens of single nest and queens of general population VII 48–63
— of Spring and Autumn Queens VIII 1–12
Variability of wings (with plate) V 407–22
— of, compared with termites VII 53, 59
Water-flea (Daphnia magna)
Relationship between cell and body size II 255–59
Watson, D. M. S.
Terns' eggs, co-operative study XV 294–345
Watson, K. M.
Terns' eggs, co-operative study XV 294–345
Webb, T. L.
Cancer-houses VIII 430–36
Weddle (reference)
Quadrature Formula I 275
Weight
and Age, Glasgow schoolchildren X 288–339
and Brain-weight IV 64–66
and Fertility in Rats VII 381–85
and Height, Glasgow Schoolchildren X 288–339
and Pigmentation (Elderton) VIII 340–53
— (Saunders) VIII 376, 377
of seed, influence of on plant IX 11–21, X 72–84
Vital capacity, stature and stem-length, inter-relation of XIV 316–36
See also Anthropometry
Welcker, H. (reference)
Moriori Crania XIII 338
Weldon, Walter (reference) V 3–7, 12
Weldon, W. F. R.
Albinism and Mendel's Laws III 107, 108
Asters II 113, 114
Bateson's revision of Mendel's theory II 286–98
Celandine, Organic correlation of Ficaria ranunculoides I 125–28
Clausilia itala, selection in III 299–306
— laminata, Natural selection in I 109–24
Cranial measurements, Naqada (reference) I 411, 453, 464
History of Correlation (reference) XIII 41, 42
Inheritance, Mathematical Theory VI 80–93
— Mendel's Law in Peas I 228–54
Memoir of his life V 1–50

Weldon, W. F. R. (*cont.*)
Memoirs by, List of V 51, 52
Mendelism II 44–55
Mice, Inheritance of sex-ratio and size of litter V 436–49
Mice, Records of matings XI, Appendix 1–60
Origin of Species, De Vries on I 365–74
Snails, selection in I 109–24, III 299–306
Weldon, W. F. R. and K. Pearson
Inheritance in Phaseolus vulgaris II 499–503
Wheat
influence of environment on I 367
Whitaker, L.
Law of Small Numbers, Poisson X 36–71
White, J. D. C. and M. Greenwood
Opsonic Index VI 376–401, VII 505–30
— — (reference) VII 532–41, VIII 203–24
Whitechapel Crania
Variation and Correlation in, comparison with other races (with plates) III 191–244
Cranial Measurements, compared with Moorfields V 92–104
— — — Moriori XI 95–133
— — — Negro VIII 302–24
— — — Tibetan XIV 208–49
Cranial Types, comparison of contours with Modern English VIII 131–33
— — contours (plates) VIII 143–47
Whitehead, H.
Variation in Moschatel II 108–13
— — (reference) VII 218
Whiteley, M. A. and M. A. Lewenz
Variability and correlation of human hand I 345–60
Comparison of results for hand measurements and finger-prints (reference) X 451
White-lock Family
X 197–200, (Barton) XIII 347–49
See also Piebalds
Whiting, M. H.
Criminal Anthropometry XI 1–37
Whitney Family V 80–85
Inheritance of sex-ratio in VI 110–11
Whooping-cough
and pigmentation (Macdonald) VIII 25–39
— (Saunders) VIII 360–65
Wiersma, E.
Data on inheritance of psychical characters V 460–69
Willis, J. C. (reference)
Age and Area theory and Natural Selection XV 89–108
Windle, B. C. A. (reference)
Sesamoids XIII 368, 369, 372, 374
Wishnewsky
Cephalic Index and Stature of Russians XV 237
Wolff, E.
Terns' Eggs, co-operative study XII 308–54
Woods, F. A.
Rabbit-breeding II 299–306
Sex-ratio in man, inheritance of V 73–78
(reference) Notice of Book on Mental and Moral Heredity in Royalty VI 445
Wright, Alexandra, A. Lee and K. Pearson
Wasps V 407–22
Wright, Almroth (reference)
Opsonic Index VI 376–401, VIII 204–24
Würtemberg
Cranial Measurements, compared with English, Whitechapel III 207–15
— — — Moriori XI 95
— — — Negro VIII 302–14, 321

Young, A. W.
Small samples, standard deviations of samples of two or three XI 277–80
Young, A. W. and K. Pearson
Probable Error of Contingency Coefficient, without Approximation XI 215–30
— — — — Corrigenda XII 259, 260
Product Moments, Normal Correlation surface of 2 variates XII 86–92
Young, A. W., H. E. Soper, B. M. Cave, A. Lee and K. Pearson
Small samples, distribution of Correlation Coefficient in XI 328–413
Young, M. and A. A. Mumford
Inter-relationships of physical measurements and vital capaci XV 109–33
Yule, G. U.
Age and Area theory and Natural Selection (reference) XV 89–108
Anemone, variation of number of sepals in I 307–9
Ancestral Heredity (reference) II 228, 229
Association of Attributes II 121–34
— Theories of (reference) IX 159–316
Convenient means of drawing curves to various scales III 469
Correlation Formulae (reference) VIII 109–22
Crania, Whitechapel (reference) III 205
Death-rates, Local I 384
Goodness of Fit, χ^2 test for fourfold tables (reference) XIV 189–91
History of Correlation (reference) XIII 45
Notice of paper on Correlation treated by a new notation VI 124
— — on Inheritance and Mendelism V 481
Reply by Karl Pearson to his Criticisms (reference) V 470–76
Variate Difference Method, reply to criticism *re* swamping by periodic term (reference) XIV 282–309

Ziegler
and Galton's Law of Ancestral Inheritance V 184
Zulus
Cranial measurements, and comparison with other races VIII 292–337



The L'Estrange Portrait. (*By Permission.*)

Fig. 2. Miniature of Sir Thomas Browne in the Buccleuch Collection. Recently on loan to the Victoria and Albert Museum. (By permission.)

Fig. 1. Plumbago Drawing of Sir Thomas Browne in the National Portrait Gallery, attributed to Loggan. (By permission.)

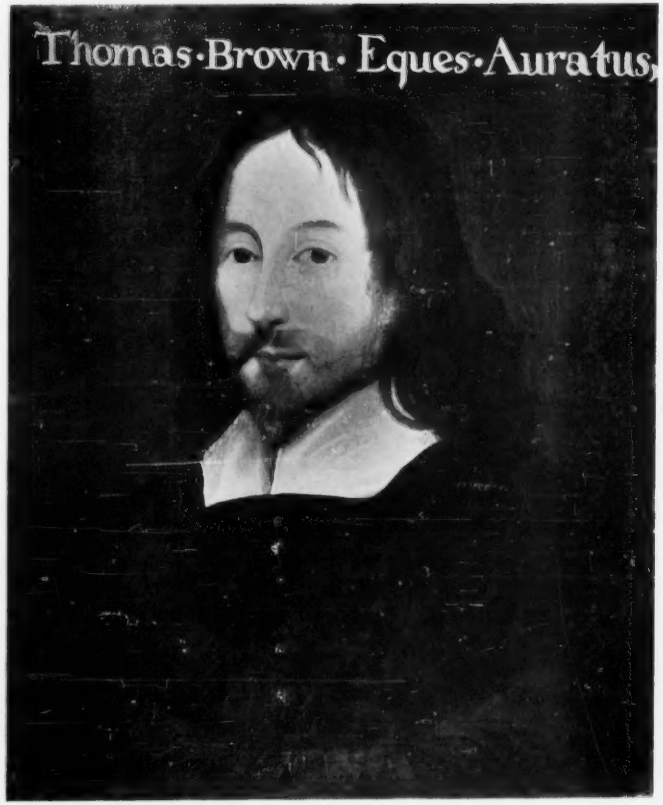


Fig. 3. Painting of Sir Thomas Browne at the Royal College of Physicians. (By permission.)

Fig. 4. Painting of Sir Thomas Browne at the Bodleian Museum, Oxford. (By permission.)

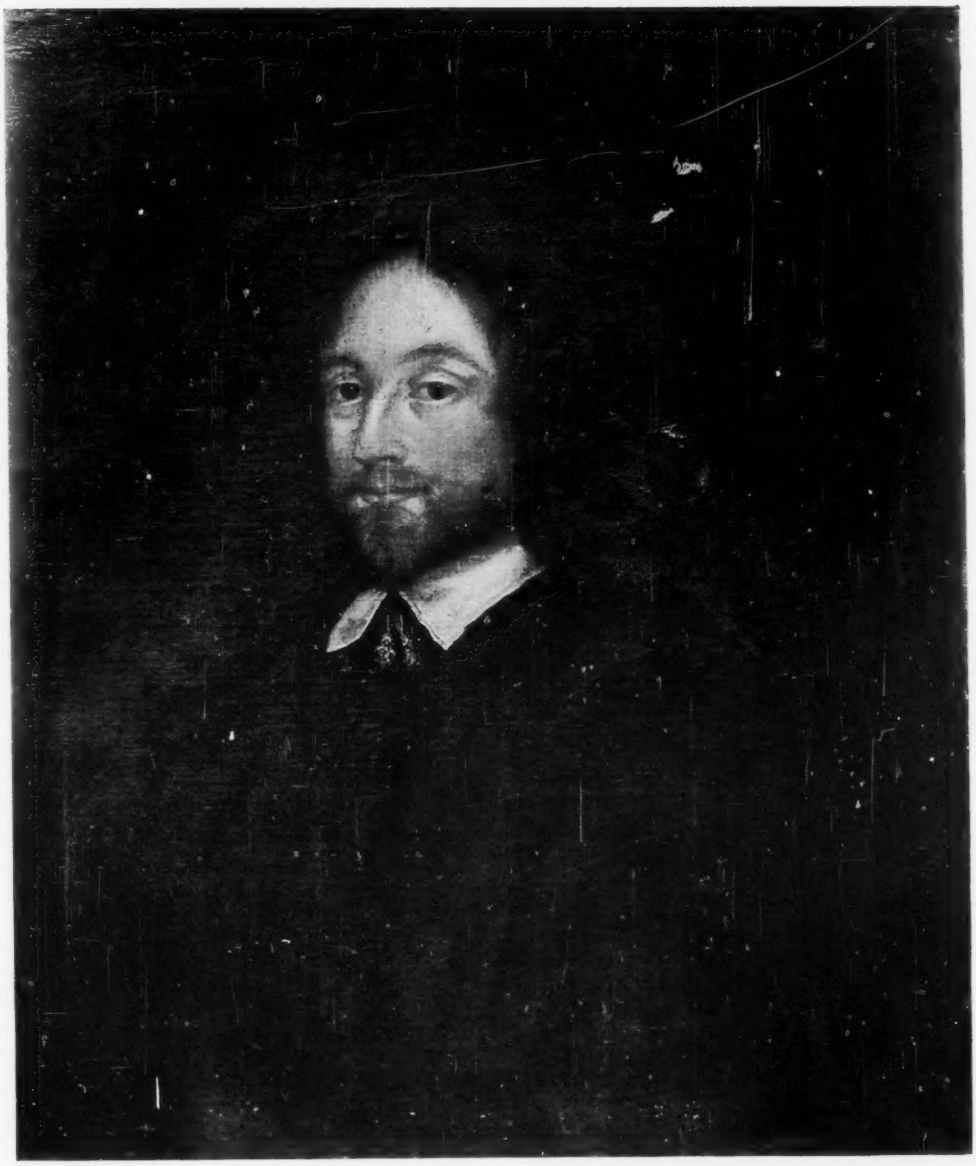


Fig. 5. Painting of Sir Thomas Browne in the Vestry, St Peter Mancroft, Norwich. (By permission.)

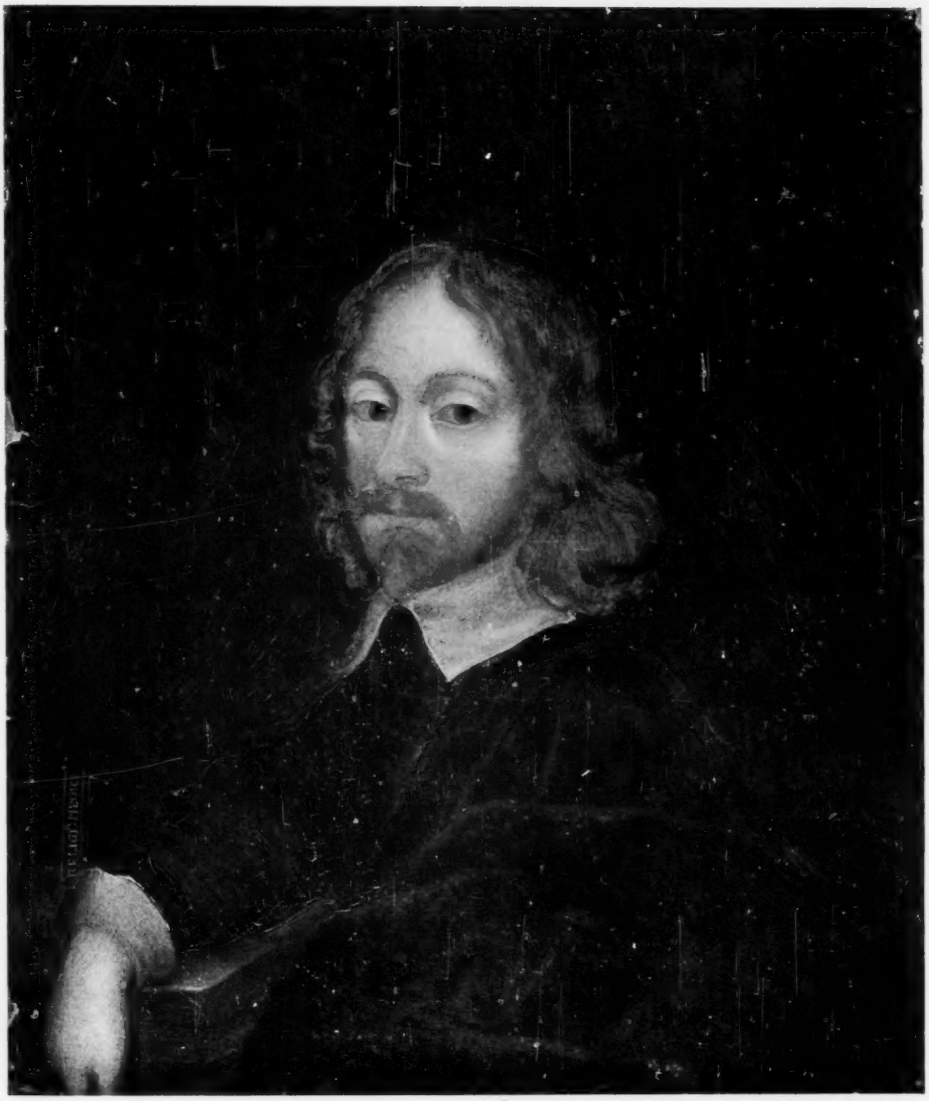


Fig. 6. Painting of Sir Thomas Browne at Gunton Hall, Norfolk. (By permission.)

Fig. 7. The alleged Portrait of Sir Thomas Browne at Sherborne House. (By permission.)

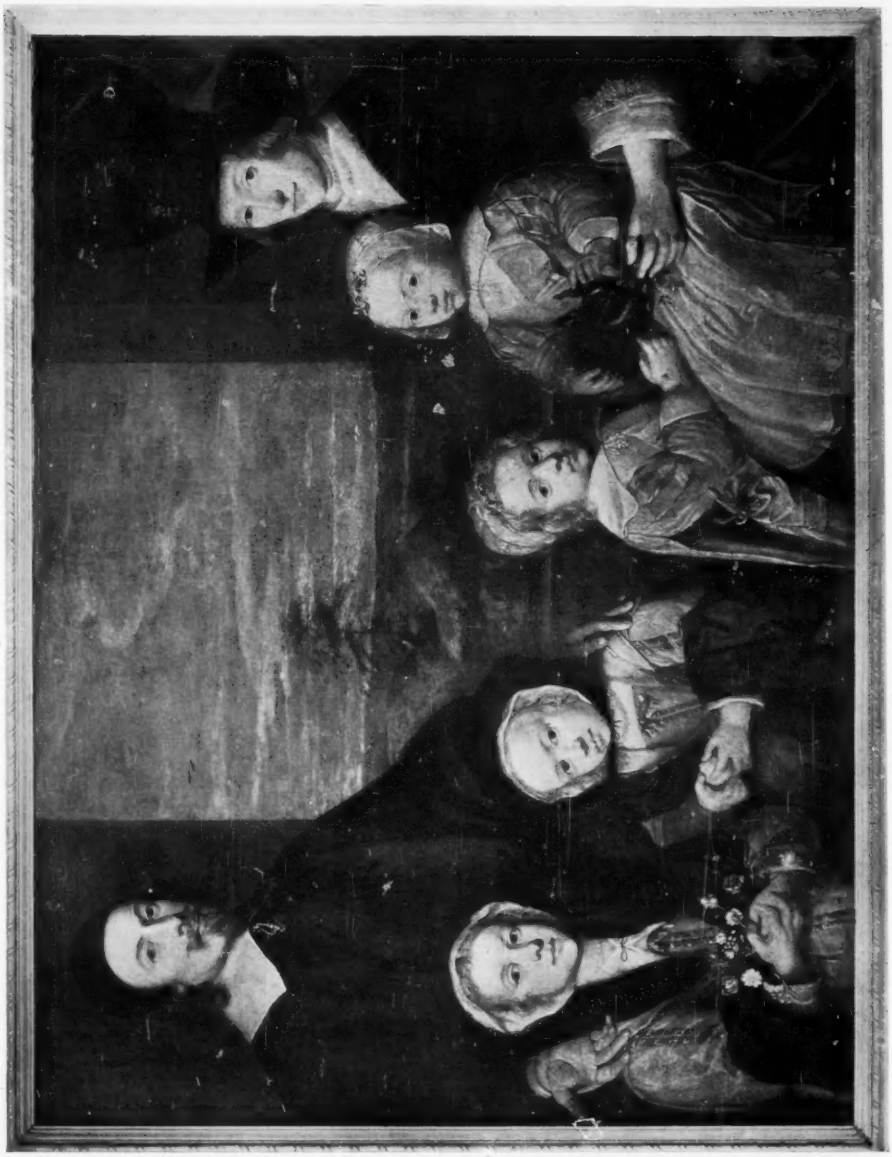


Fig. 8. The "Dobson" Group in the possession of the Duke of Devonshire. (By permission.)

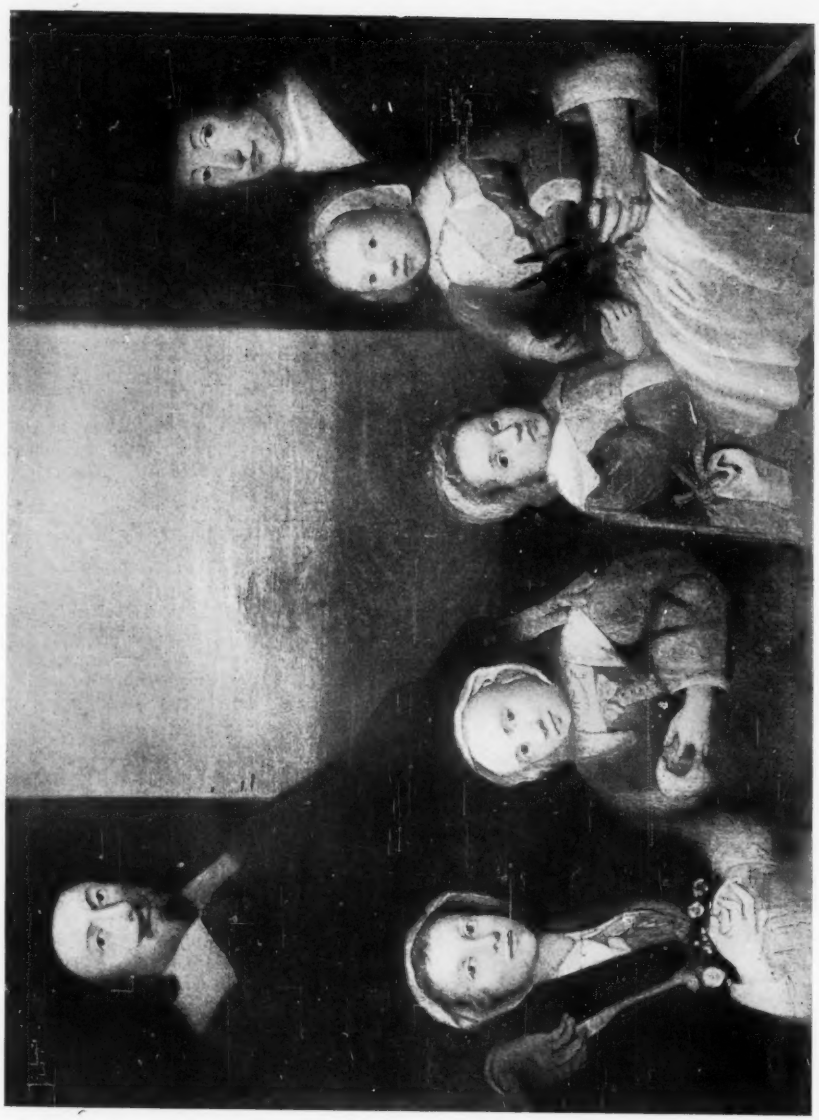


Fig. 8 (bis). The "Dobson" Group from a photograph in the possession of the Norwich Castle Museum. (By permission.) The original of this photograph as will be seen on comparison with Plate VI must differ very considerably from the Duke of Devonshire's picture.

Fig. 9. Engraving issued as a portrait of Sir Thomas Browne with the 5th Edition of the *Pseudodoxia*, 1669.

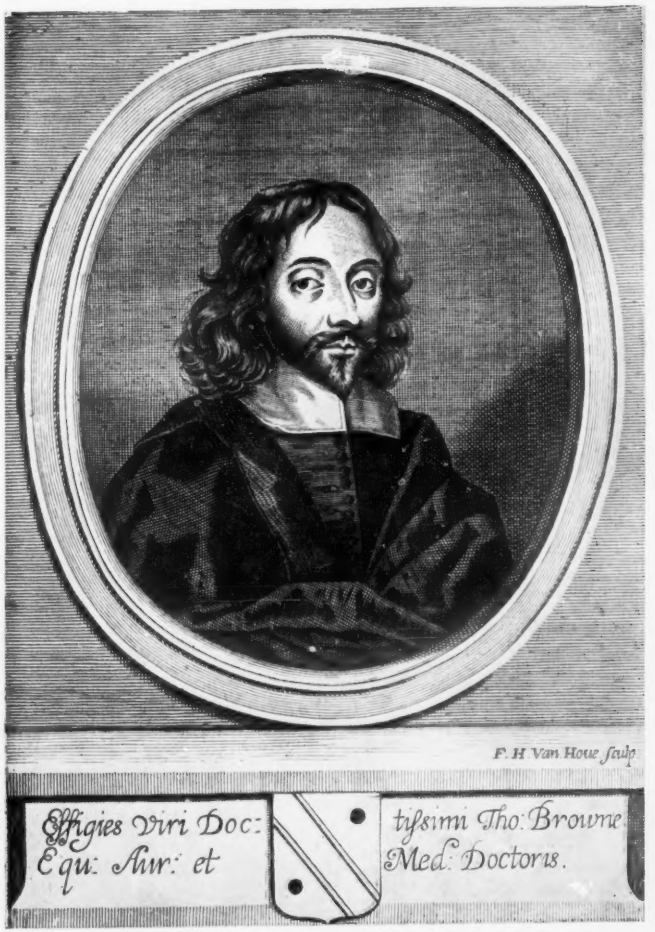


Fig. 10. Van Hove's engraving of Sir Thomas Browne issued with the 6th Edition of the *Pseudodoxia*, 1672.

Fig. 11. Van der Banck's engraving of Sir Thomas Browne issued with the *Certain Miscellany Tracts* in 1683 and 1684.

Fig. 12. Robert White's engraved portrait of Sir Thomas Browne issued with the *Collected Works*, 1686.

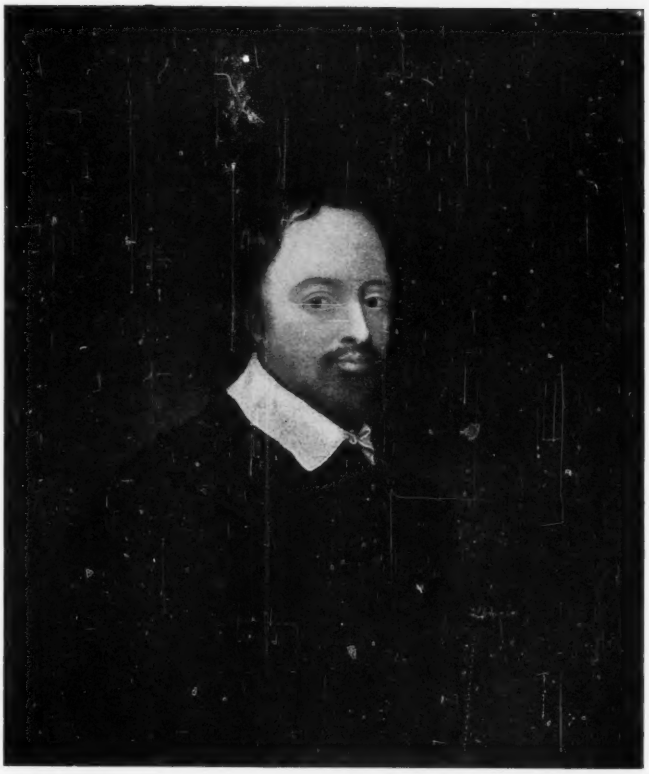


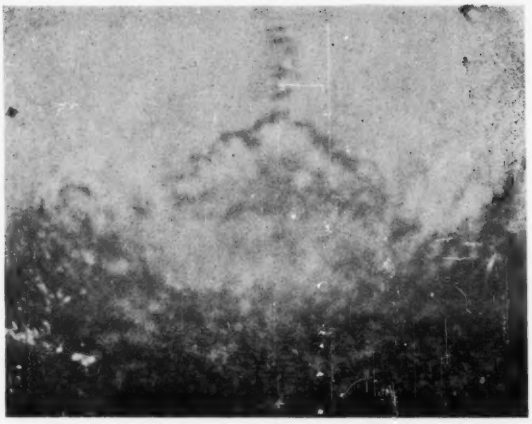
Fig. 13. A Painting said to be of Sir Thomas Browne in the Wellcome Historical Medical Museum. (By permission.)

Zoust pinx.
Shakespeare
Ob: A.D. 1616. Ætat: 53.
Done from a Capital Picture in the Collection of I. Wright Painter in Covent Garden.
I. Simon fe. et ex.

Fig. 14. The Simon mezzotint of the so-called Soest portrait of Shakespeare. To indicate that the Wellcome portrait of Sir Thomas Browne is what passes as the Soest Shakespeare.

Fig. 15. Drawing made mid-nineteenth century by Miles Edmund Cotman of the Painting of Sir Thomas Browne in the Vestry of St Peter Mancroft, Norwich. To indicate the supposed form of the hair at that time.

Fig. 16. Portrait of Dr Edward Browne, son of Sir Thomas Browne. To indicate that the receding forehead of the latter appears to have been a characteristic of the former.



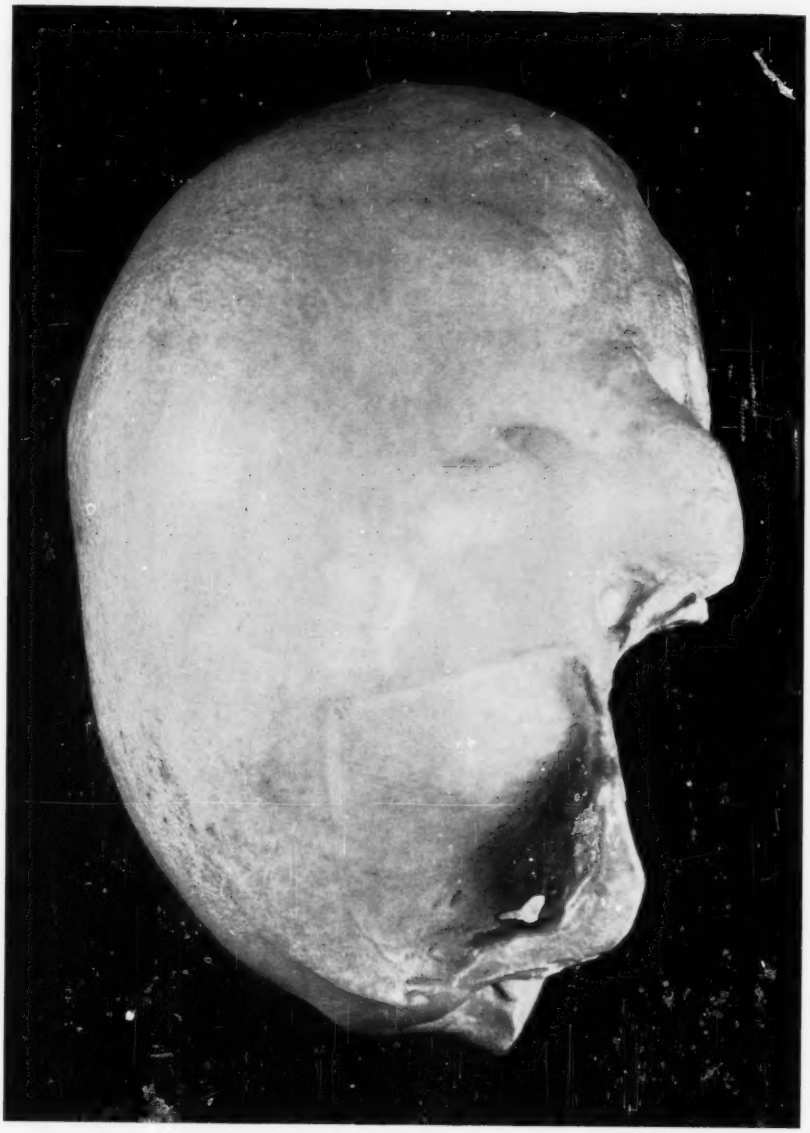
A. Photograph of skull recently re-interred.

B. Photograph of cast of 1840–1 in the Norwich Castle Museum.

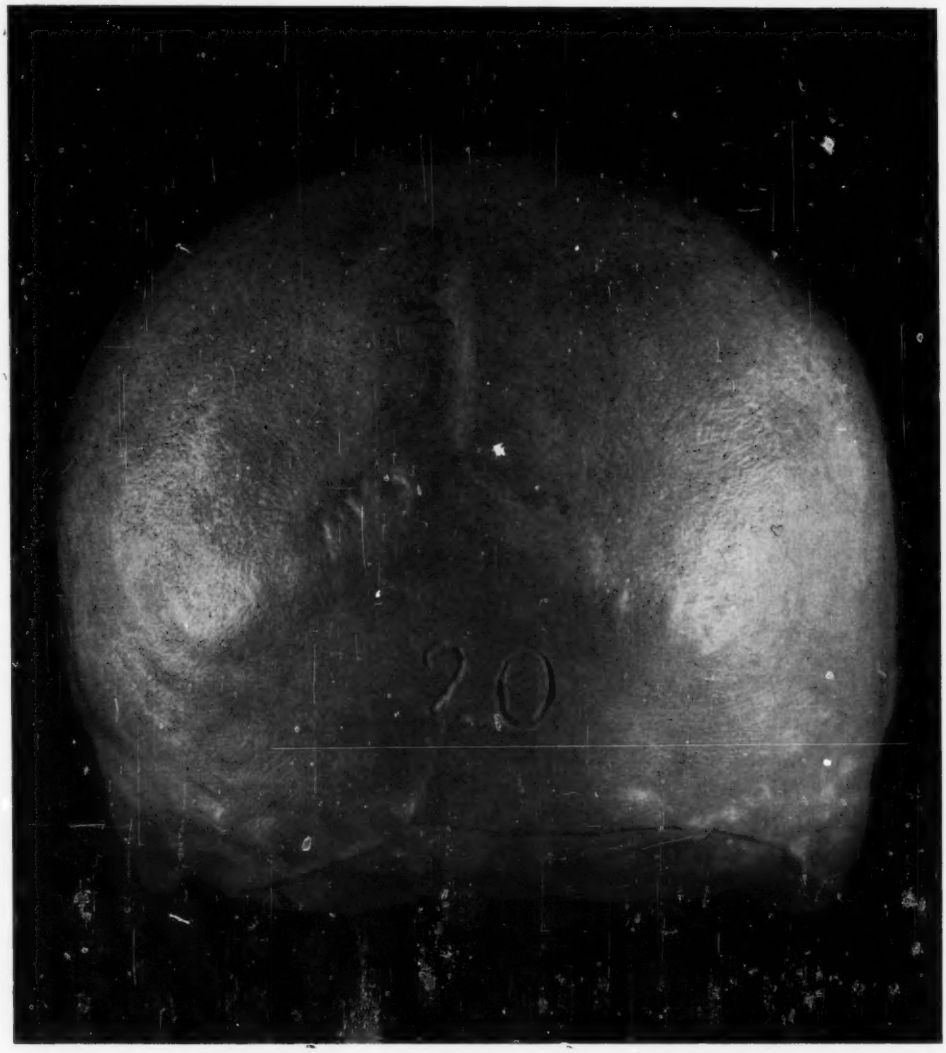
Fig. 17. Comparison between the lambdoid suture of the skull re-interred in 1922 as Sir Thomas Browne's and the same suture of the cast presented in 1841 to the Norwich Castle Museum as that of his skull. The coffin is known to have been opened in 1840.

M. S.
Hic situs est
THOMAS BROWNE M.D. et Miles
Ao 1605 LONDINI natus
Generosa Familia apud UPTON
In agro CESTRIENSI oriundus
Schola primum WINTONIENSI postea
In Coll. PEMBR.
Apud Oxonienses bonis literis
Haud leviter imbutus.
In urbe hac NORDOVICENSI medicinam
Arte egregia, & felici successu professus.
Scriptis quibus inscribi. RELIGIO MEDICI
Et PSEUDODOXIA EPIDEMICA aliisque
Per Orbem notissimus.
Vir Prudentissimus, Integerrimus, Doctissimus,
Obiit Octob. 19. 1682.
Pie posuit mestissima Conjux
Da Doroth. Br.
Near the foot of this Pillar lies Sr Thomas Browne Kt and
Doctor in Physick, Author of Religio Medici, and other
Learned Books, who practic'd Physick in this City 46 years,
died Octo 19. 1682, in the 77th year of his Age. In Memory of
whom, Dame Dorothy Browne, who had been his affectionate
Wife 41 years, caused this Monument to be erected.

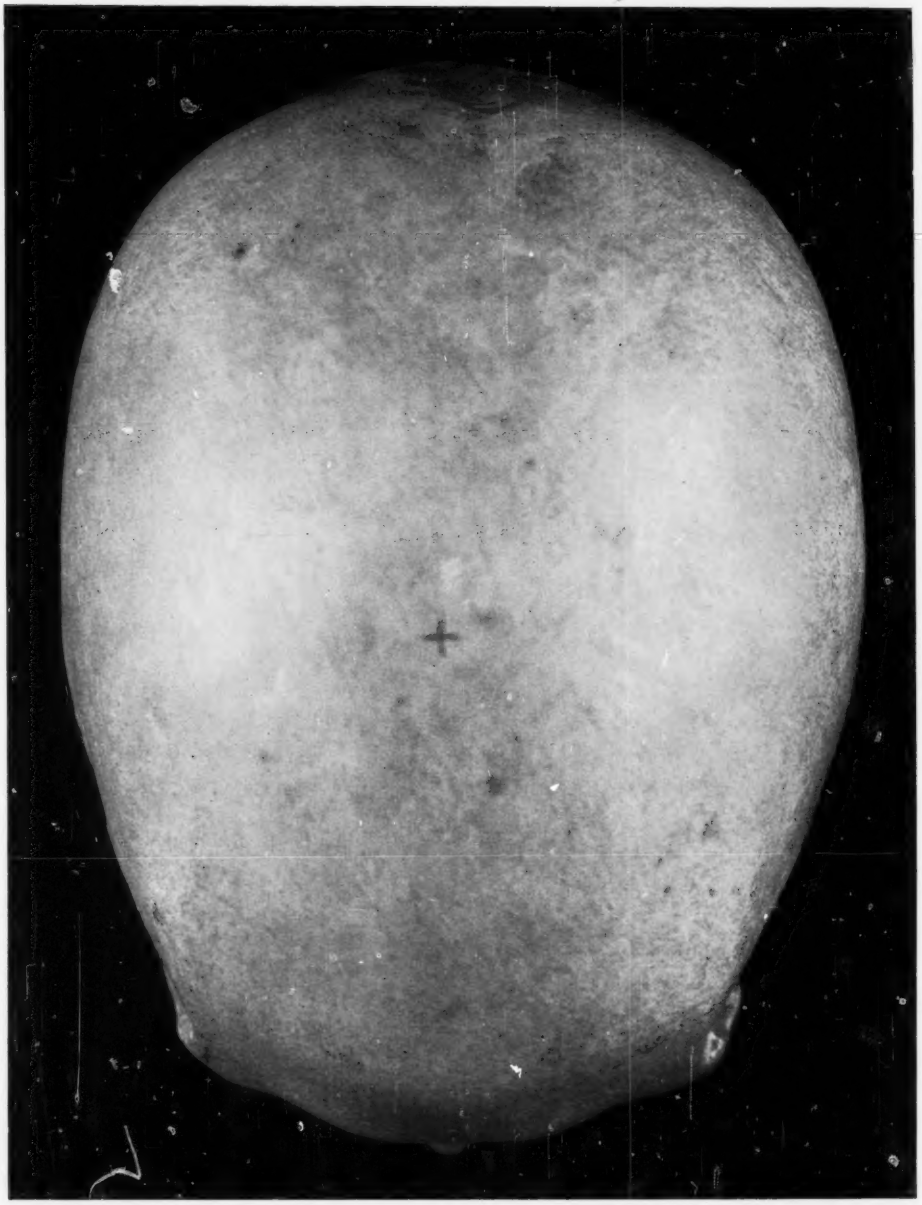
The Memorial Tablet to Sir Thomas Browne in St Peter Mancroft, Norwich. Reproduced from the engraving in Vol III of Wilkin's edition of the *Collected Works*, 1838.



Cast of Skull taken in 1840–1. *Norma lateralis* (left profile.)

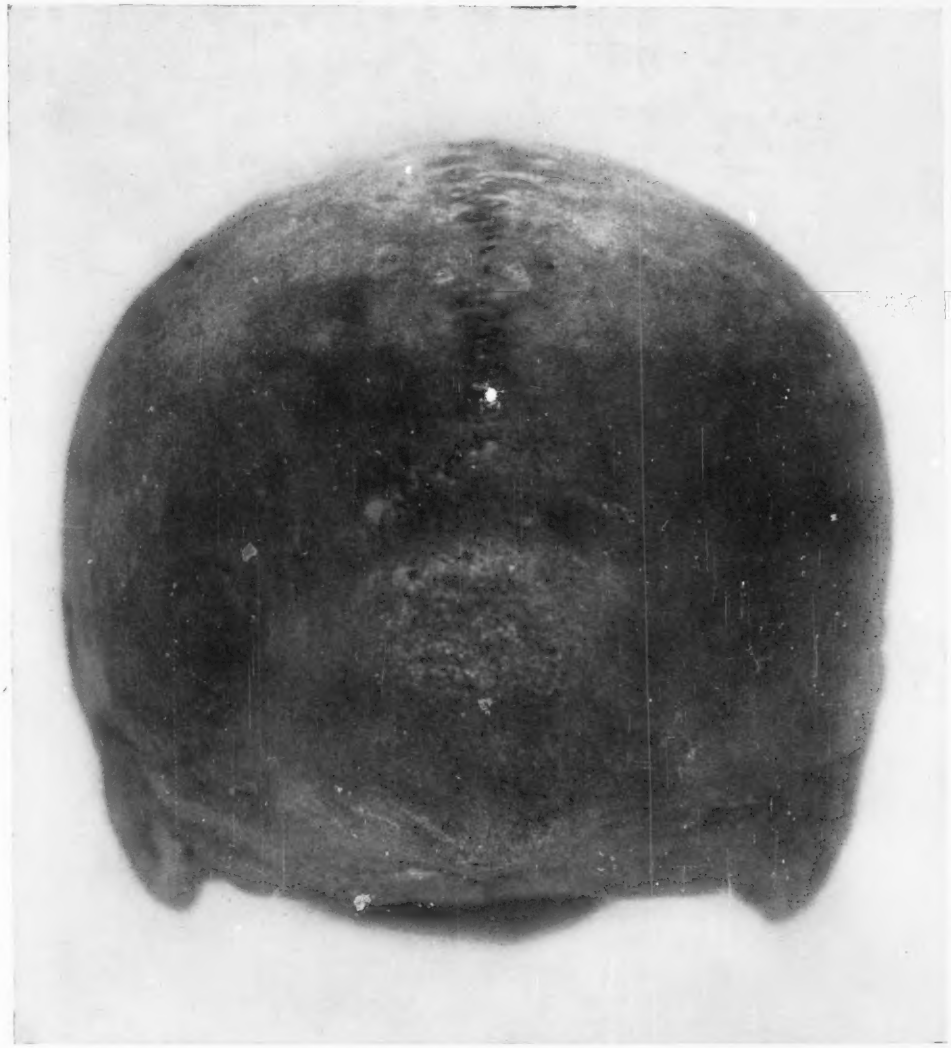


Cast of Skull taken in 1840–1. *Norma occipitalis.*



Cast of Skull taken in 1840–1. *Norma verticalis.*

Sir Thomas Browne's Skull. *Norma lateralis* (left profile).

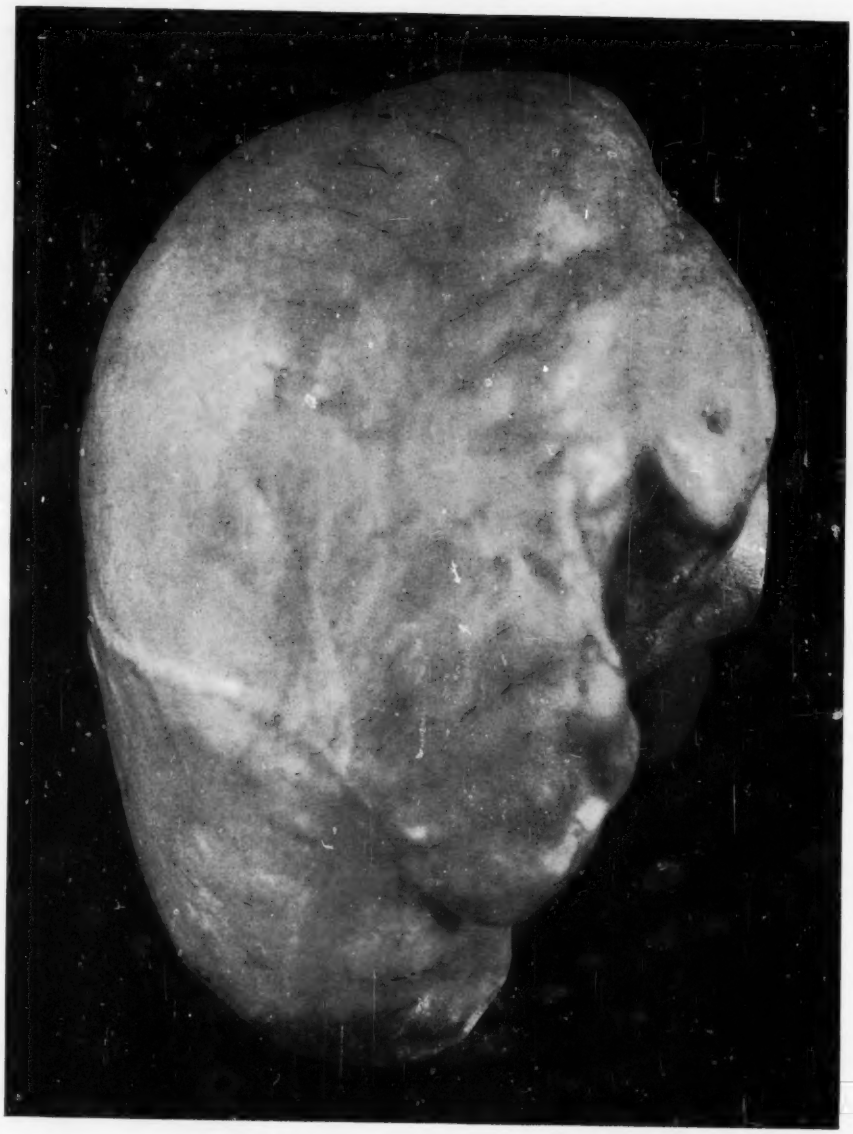


Sir Thomas Browne's Skull. *Norma occipitalis.*

Sir Thomas Browne's Skull. *Norma verticalis.*

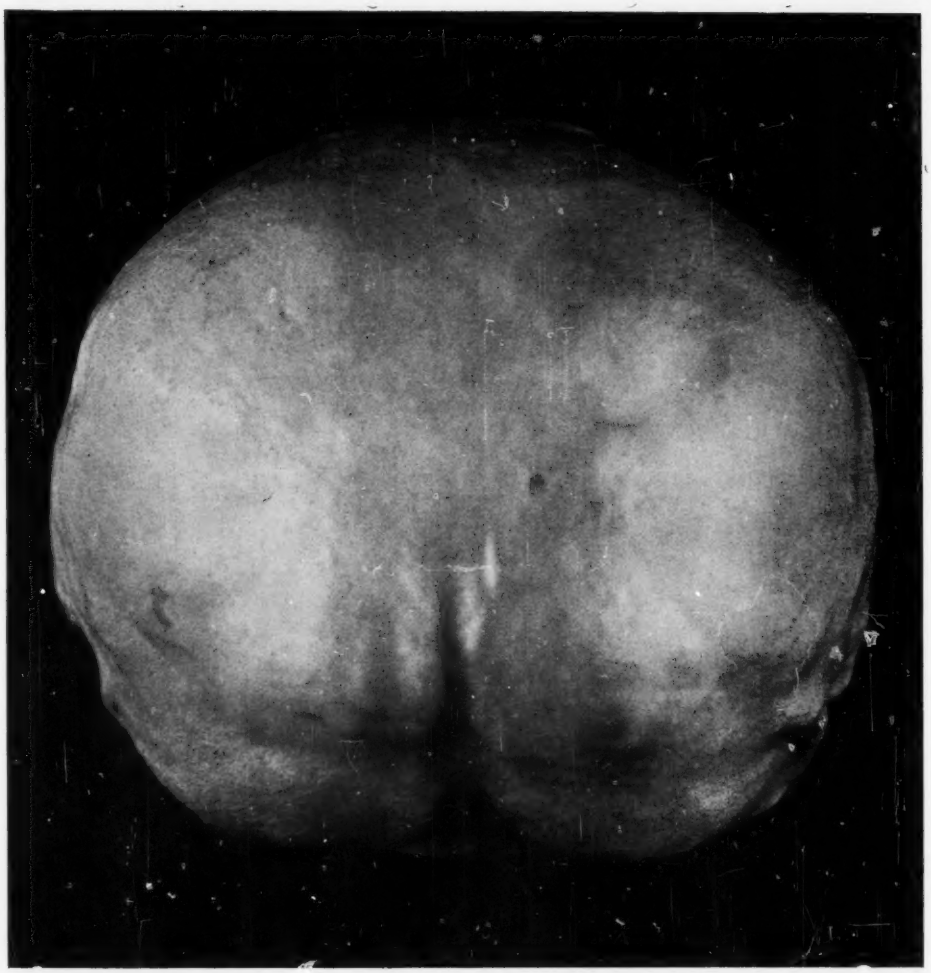
Sir Thomas Browne's Skull. *Norma facialis.*

Sir Thomas Browne's Skull. *Norma basalis.*

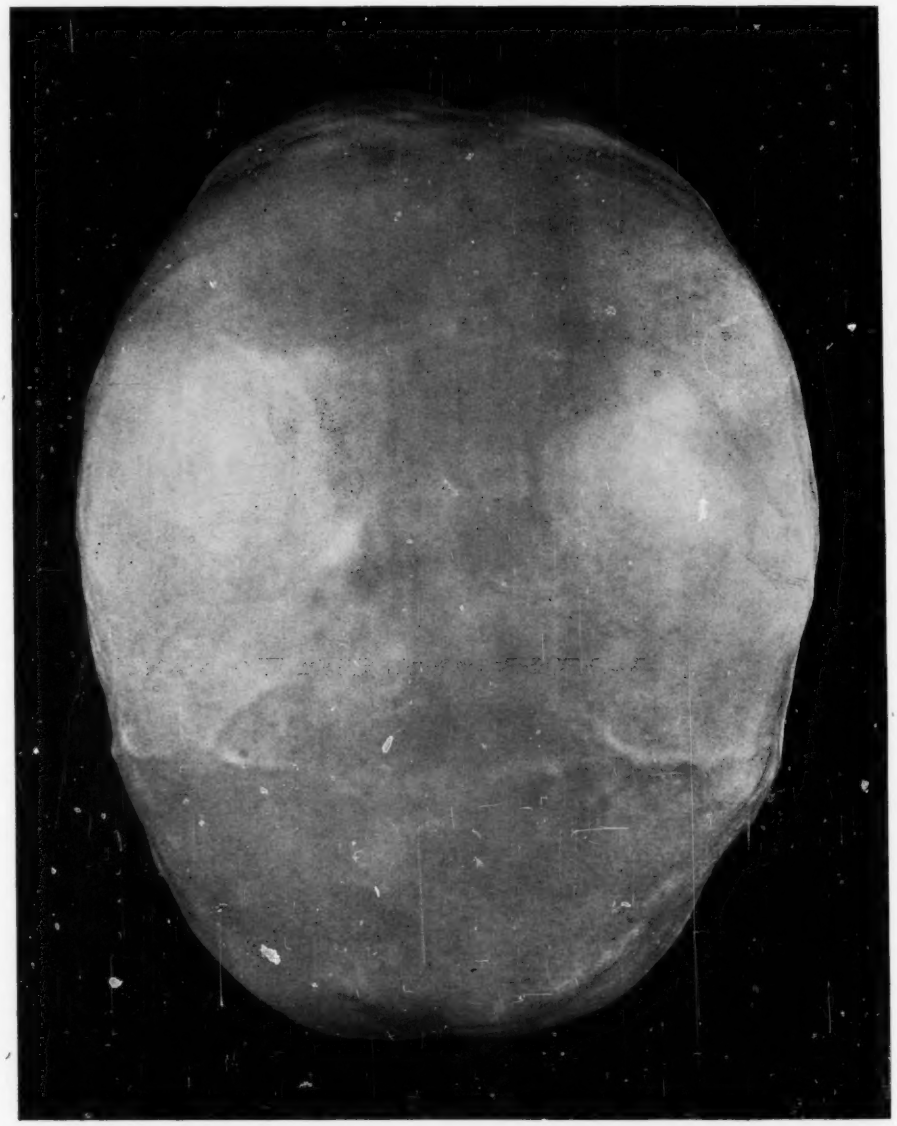


Endocranial Cast of Sir Thomas Browne's Skull. *Norma lateralis* (left profile).

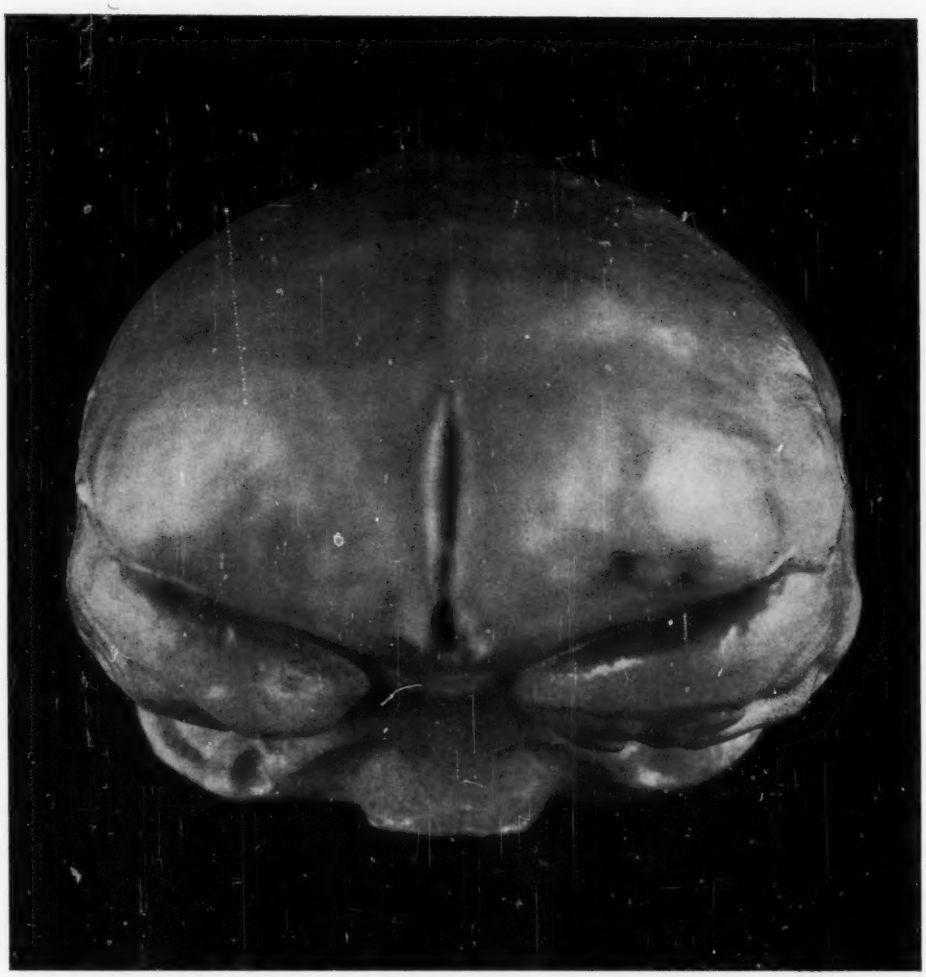
Endocranial Cast of Sir Thomas Browne's Skull. *Norma lateralis* (right profile).



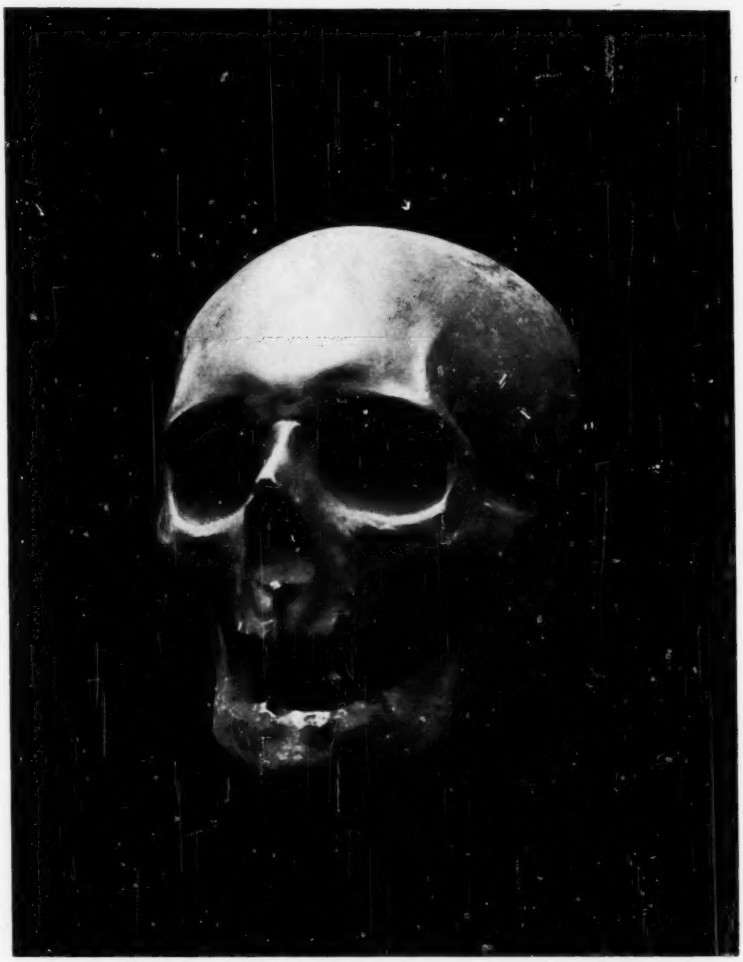
Endocranial Cast of Sir Thomas Browne's Skull. *Norma occipitalis.*



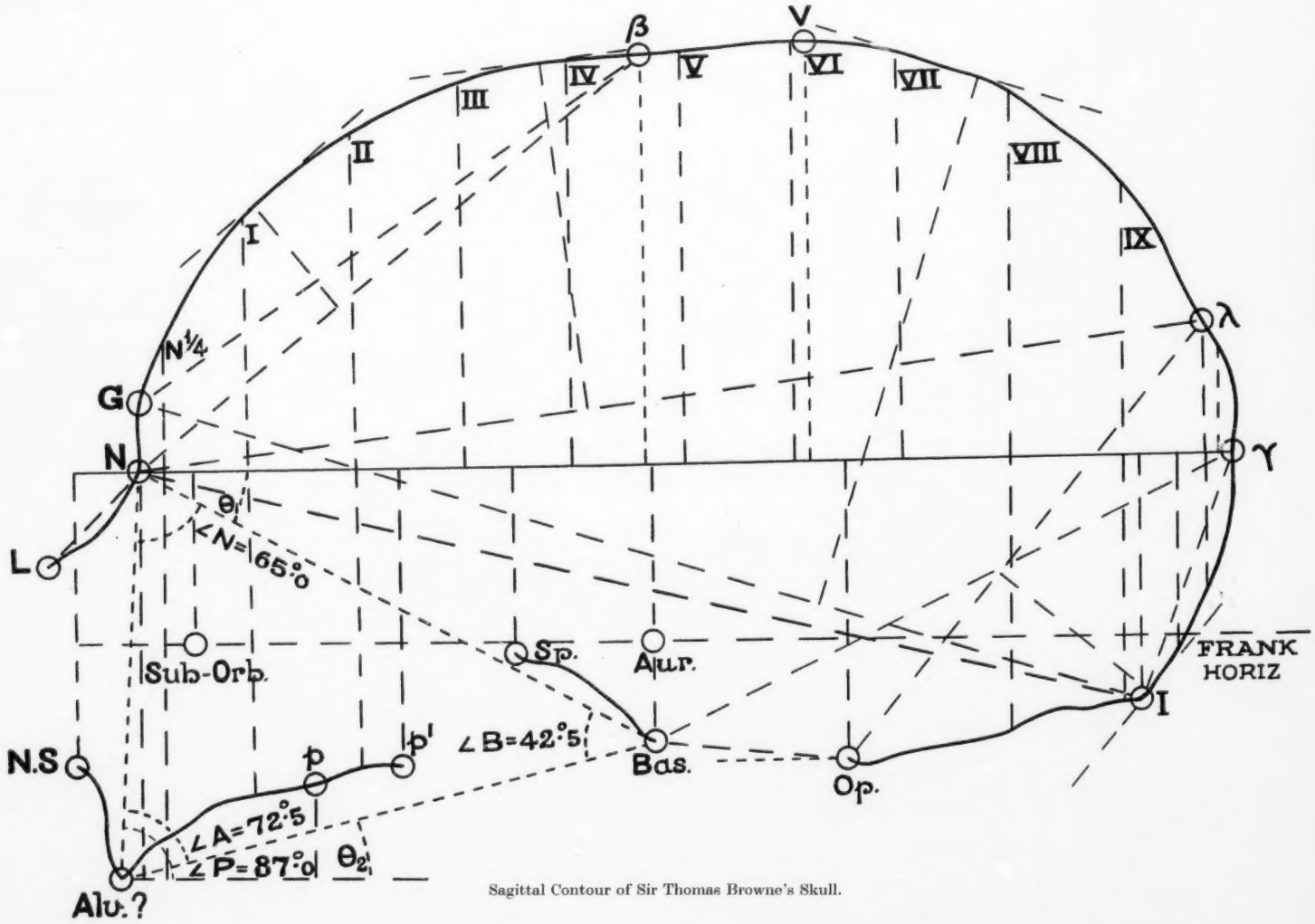
Endocranial Cast of Sir Thomas Browne's Skull. *Norma verticalis.*



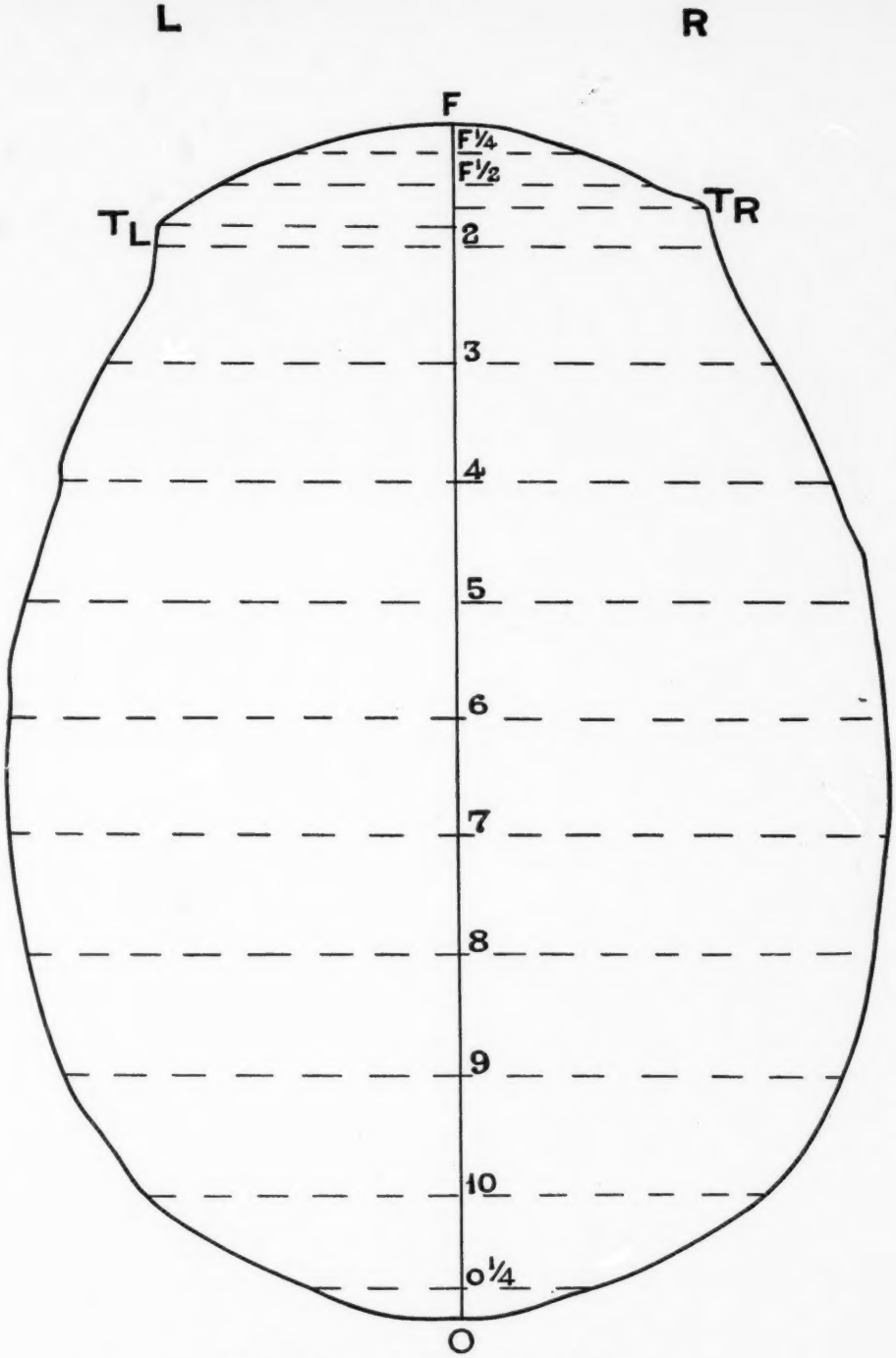
Endocranial Cast of Sir Thomas Browne's Skull. *Norma frontalis.*



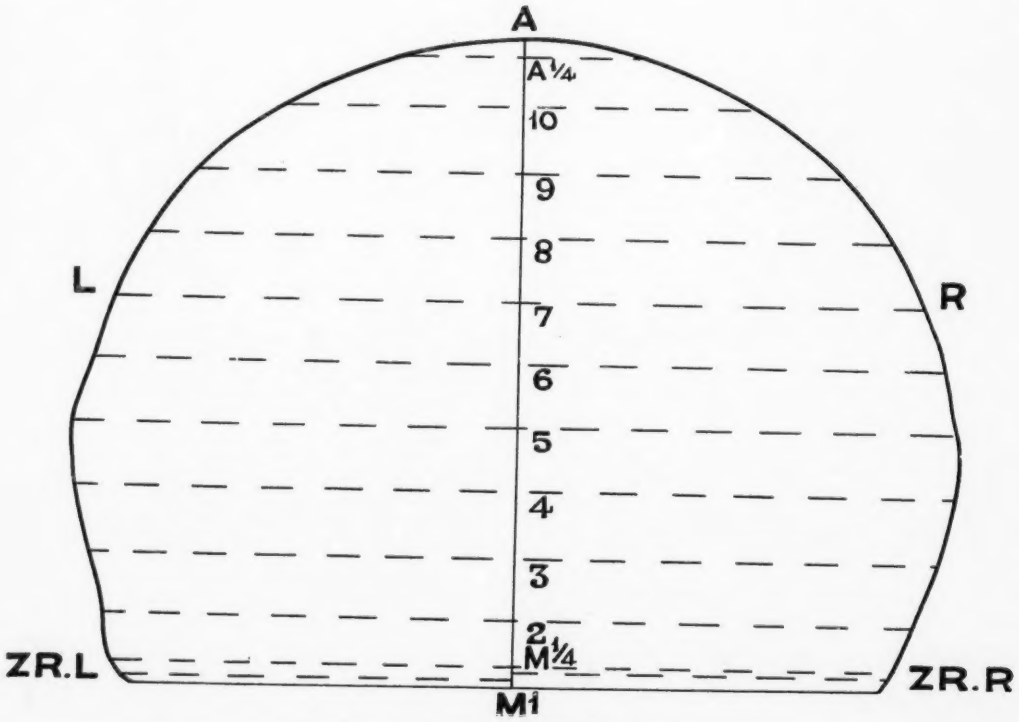
Cast of Sir Thomas Browne's Skull taken in 1922, and arranged three-quarter face to correspond in aspect with the Buccleuch miniature: see Plate I.



Sagittal Contour of Sir Thomas Browne's Skull.



Horizontal Contour of Sir Thomas Browne's Skull.



Transverse Contour of Sir Thomas Browne's Skull.

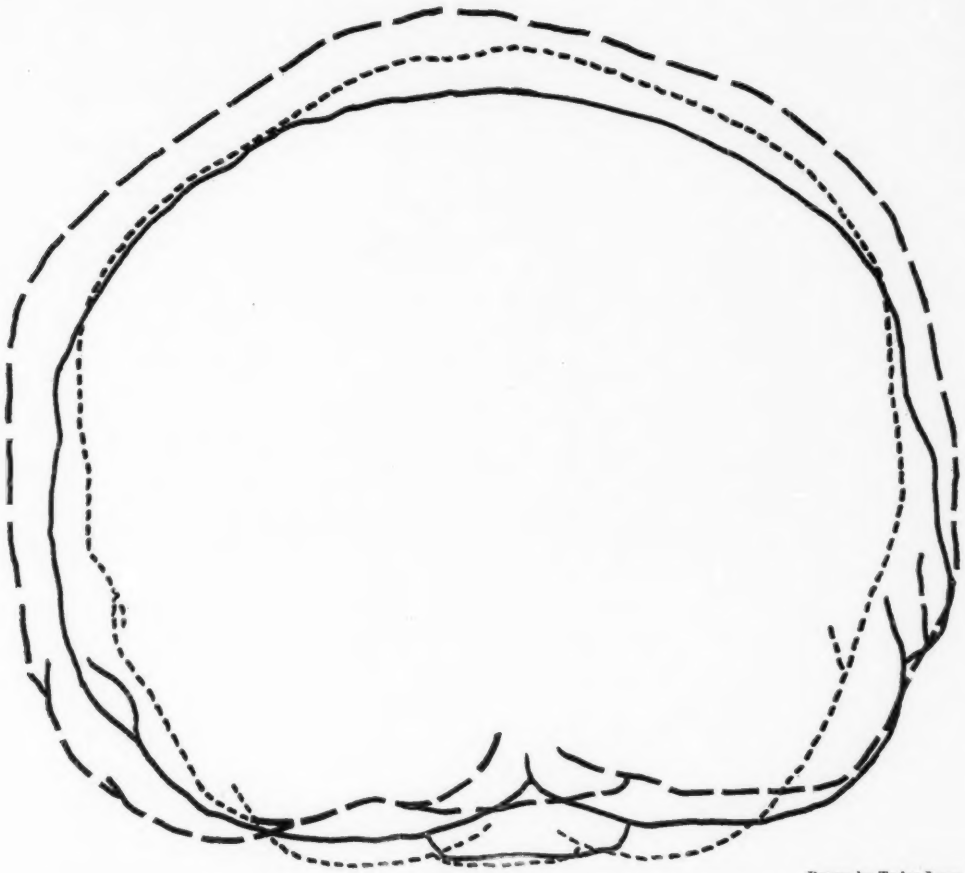
Orthographic Projections of Endocranial Casts. *Norma lateralis* (left profile). Drawn by Tudor Jones

— — — — Dean Swift ——— Sir Thomas Browne - - - - English Male of average cranial capacity

Drawn by Tudor Jones

Orthographic Projections of Endocranial Casts. *Norma verticalis.*

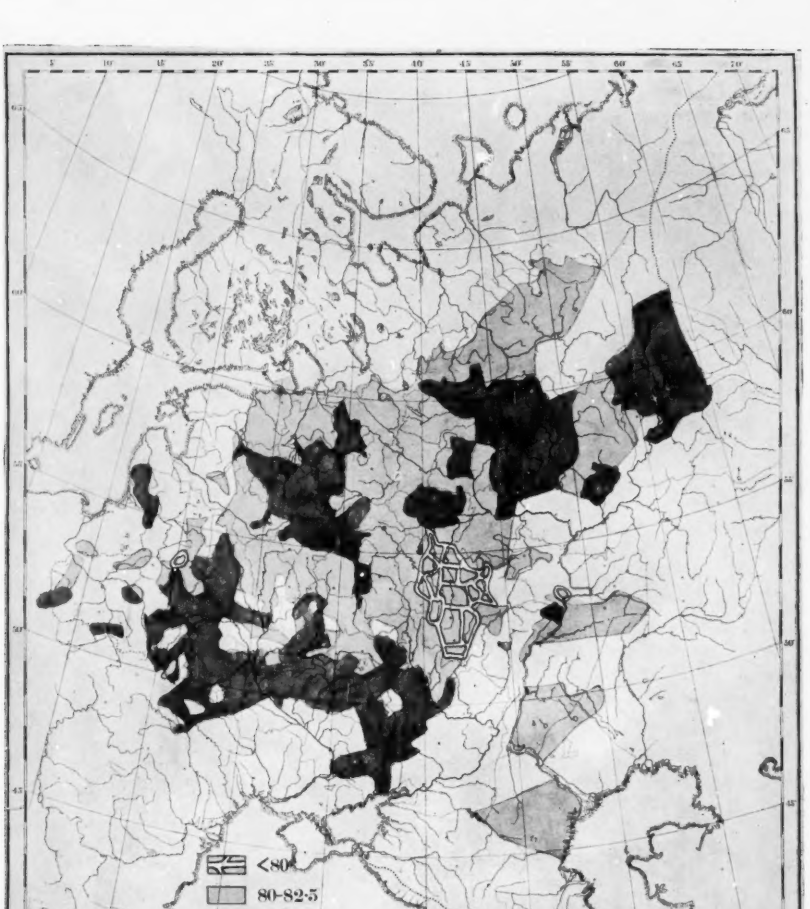
— — — — Dean Swift ——— Sir Thomas Browne - - - - - - English Male of average cranial capacity



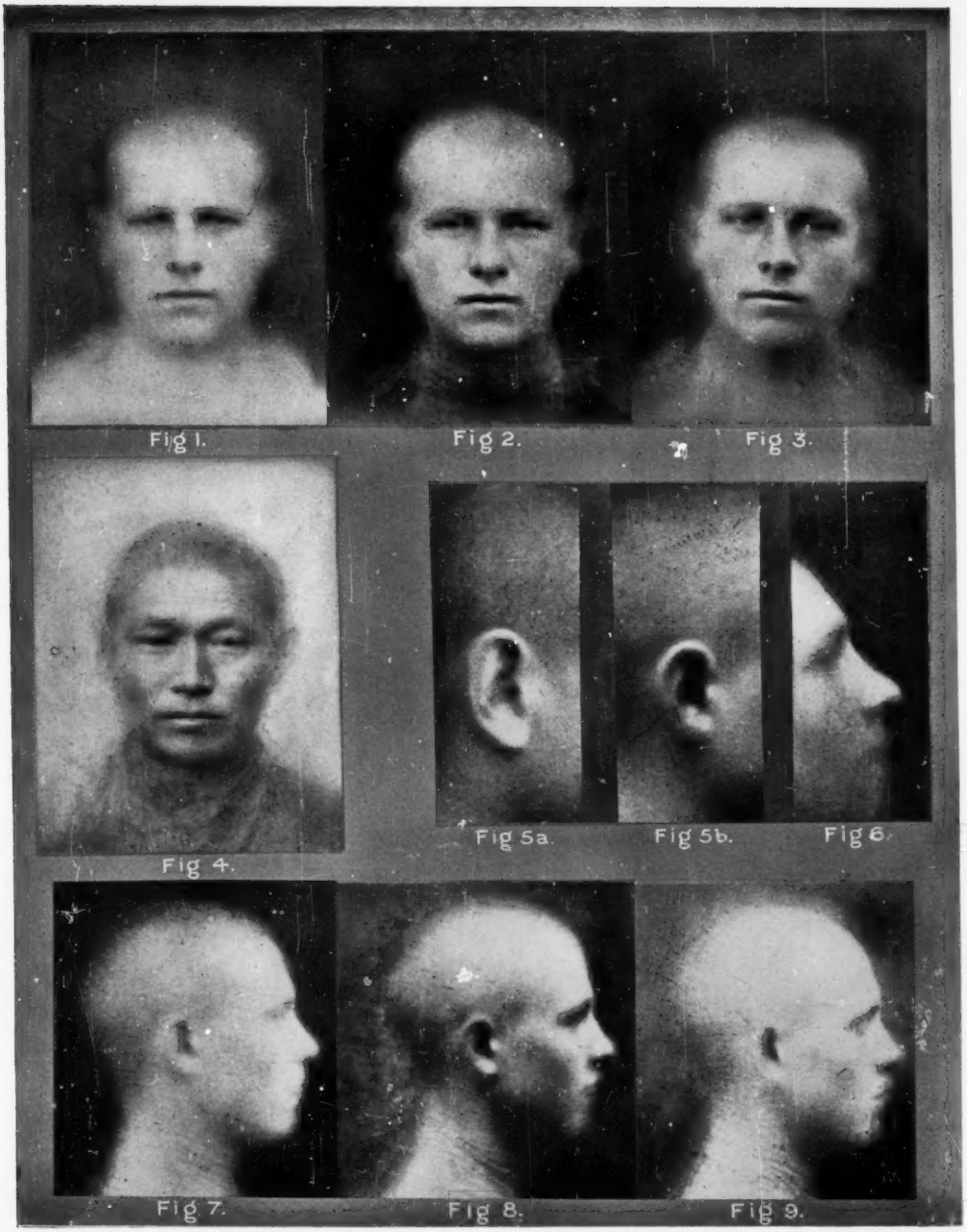
Drawn by Tudor Jones

Orthographic Projections of Endocranial Casts. *Norma occipitalis.*

— — — — Dean Swift
———— Sir Thomas Browne
- - - - - - - - - - English Male of average cranial capacity

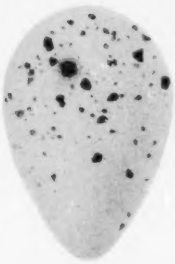


Map of European Russia, indicating the dolichocephalic, mesocephalic and brachycephalic districts.



Anthropological Composites of Russian Types

Fig. 1. Composite of 451 East Russians.
Fig. 2. Composite of 147 White Russians.
Fig. 3. Composite of 140 Little Russians.
Fig. 4. Composite of 25 pure Mongols.
Fig. 5*a*. Composite of Ear of Great Russians.
Fig. 5*b*. Composite of Ear of Great Russians.
Fig. 6. Composite of Nose of Great Russians.
Figs. 7–9. Composite of Great Russians in profile.

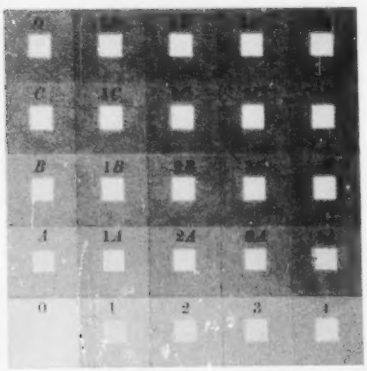


a

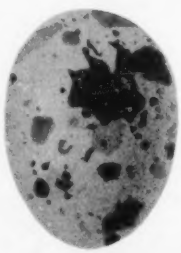
b

c

d



e

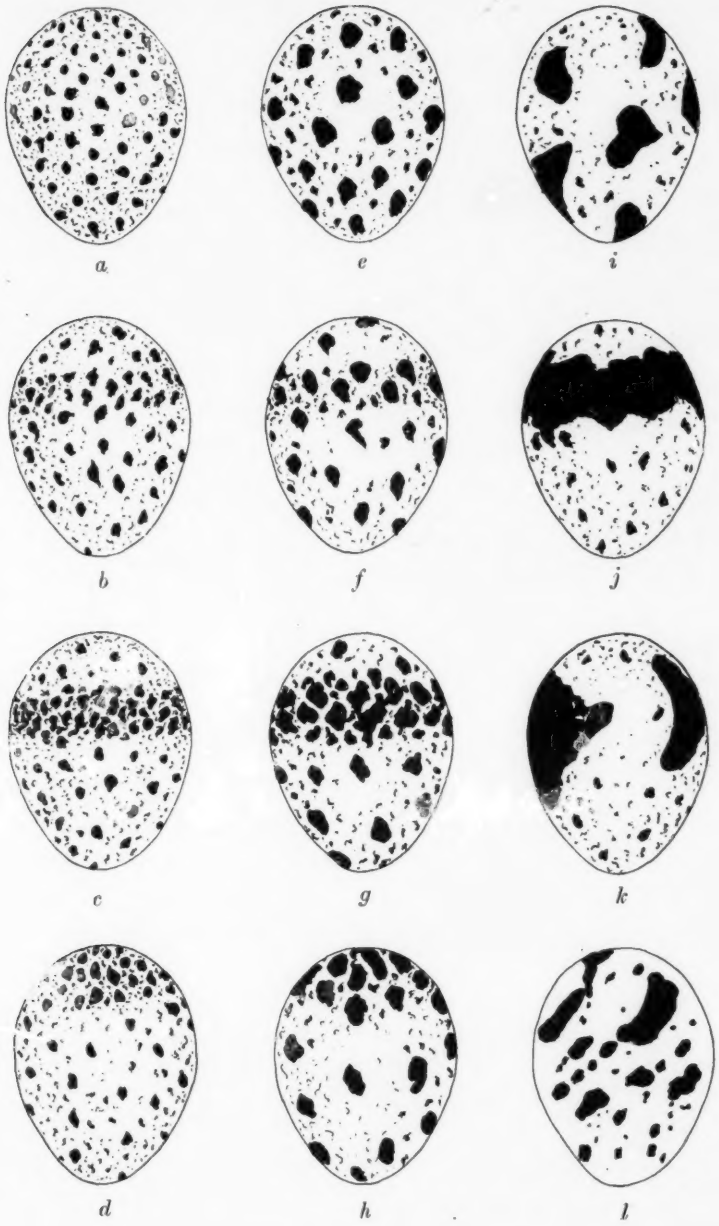


f

g

h

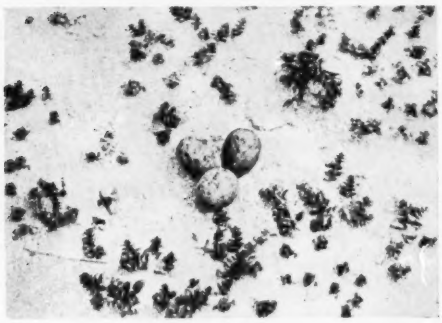
Colour Scale for Ground-Colour, and Special Eggs selected for Shape and Pigmentation.



Scale of Mottling used in 1920.

An Environment at Blakeney.

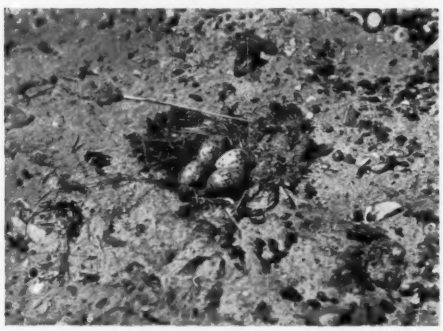
Fieldworkers measuring eggs.



No attempt at nest.

No attempt at nest.

"Bits and chips."



Some attempt at nest.

Psamma nest.

Well made nest.

Nests and their Environments from photographs of W. Rowan and Noel Karn.

Fig. (1).

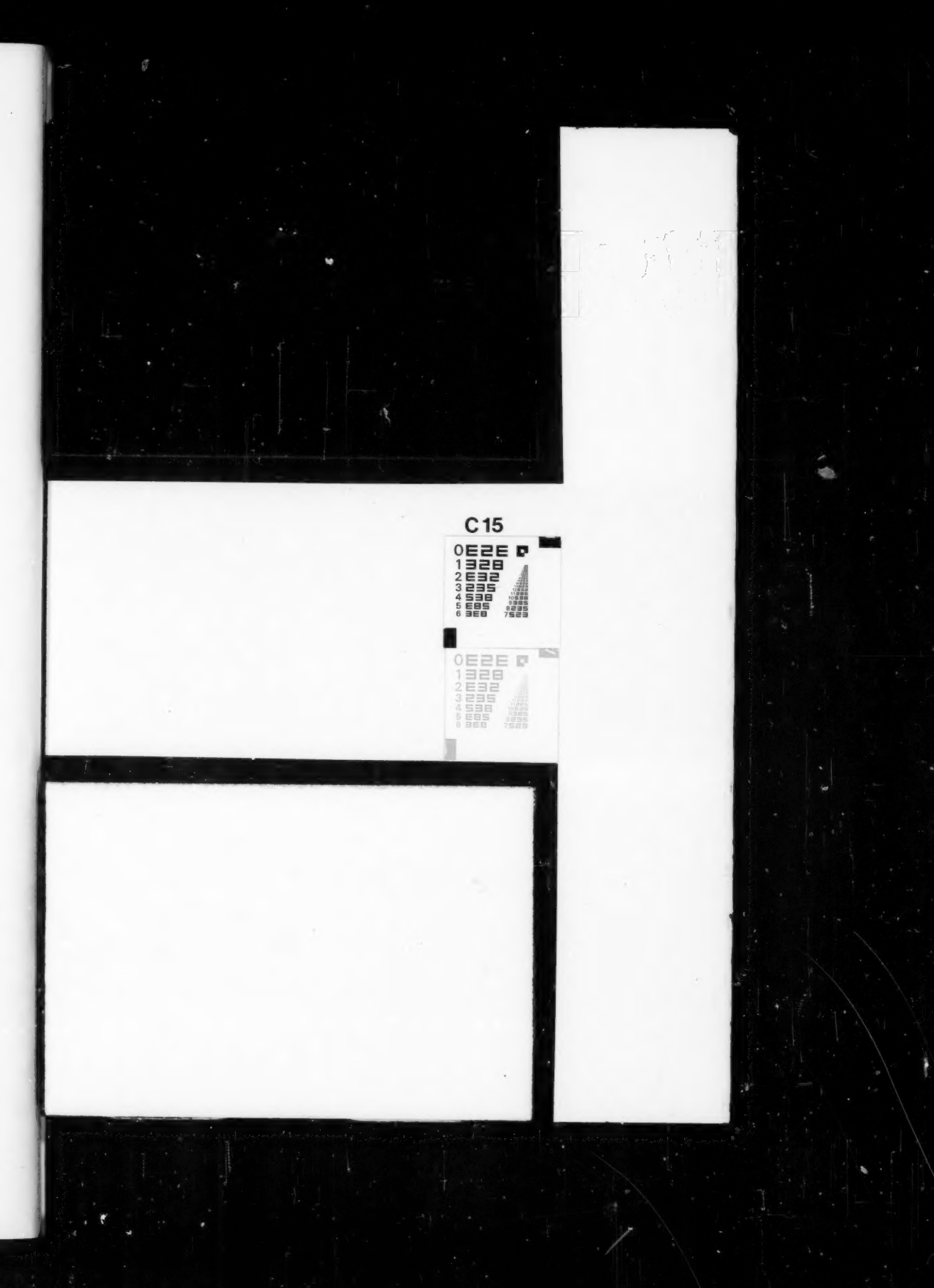
Fig. (i).

Fig. (ii).

Fig. (iii).

Fig. (i). Panoramic Photograph of "Far Point," the New Ternery, by the kindness of Prof. F. W. Oliver.
Fig. (ii). General view of Sand Dunes, the "Tern Dunes," the Old Ternery
Fig. (iii). Common Tern's Nest sheltered by prickly Saltwort (*Salsola Kali*)

Copyright Mr R. Gaze.


C15